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Infinity in Mathematics and Theology[1]

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Abstract *''Infinity'' and its derivatives are frequently used in mathematics and theology. Do these expressions denote the same thing in those distinct areas of scholarship? In this article the uses of ''infinity'' in mathematics and its uses in theology are examined and compared. One conclusion is that quite different concepts go under the heading of ''infinity.'' Although they must not be confused, there are some relations between mathematical and theological senses of infinity.*

Key words: Infinity; Mathematics; Theology; Set theory; Absolute; Cantor; Metaphysics

There are several different kinds of relations between mathematics and theology.[2] In most cases, relations are personal. So, for example, there is an article in this volume concerning possible religious motives of some Russian mathematicians.[3] Or take Georg Cantor, the Russia-born German mathematician who invented set theory and had intense contacts with Roman Catholic theologians.[4] When it comes to conceptual relations, however, bridges seem to be built much less frequently. A very prominent candidate for a bridging concept is infinity. While theology describes God as the infinite being, mathematics deals with infinite sets and numbers. Mathematics is sometimes even said to be the ''science of the infinite'' (H. Weyl).[5] Do mathematicians and theologians therefore have a common subject?[6] Do they talk about the same thing when talking about ''infinity''? In this article I want to examine some of these alleged conceptual connections.

As a starting point, I take it for granted that mathematics and theology are quite different areas of scholarship. That ''infinity'' denotes the same thing in mathematics and theology cannot be inferred from their mere usage of the same word ''infinite.''[7] Mathematics does not calculate within the realm of God's essence. And theology cannot use set theory as a means of producing new names of the unnameable. So, bluntly identifying mathematical and theological reference to infinity will surely lead into a nebular of supposed but not actual understanding. Some careful handling of concepts is required instead.

In the following, I want to scrutinize in more detail what is meant by ''infinity'' in mathematics and theology. After having discussed shortly the ''absolute infinity'' of Georg Cantor, the final section seeks relations between the kinds of infinity-talk distinguished before.

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Infinity in mathematics

Hermann Weyl was in a sense surely right when he described mathematics as the ''science of the infinite.'' Infinity plays so many roles in modern mathematics that I cannot offer something like an overview here. But there are some definitions which are so basic and general that many other kinds of mathematical infinity-talk can be put down to them. The following definition is from set theory:

> (InfSet_1) A set m is called ''infinite'' iff m is isomorphic to a proper subset of m.

This definition uses a fascinating property of infinite sets which was discovered by Austro-Czech priest, philosopher and mathematician Bernard Bolzano (1781–1848) and was later made a definition by Georg Cantor (1845–1918) and Richard Dedekind (1831–1916). It is frequently illustrated by the so-called Hilbert Hotel with infinitely many rooms. If an additional guest arrives at the fully booked hotel, every guest is made to move to the next room so that the first room is available to the new guest.

Alternative definitions of infinity use (properties of) the set of natural numbers, the most basic example of an infinite set. For example:

> (InfSet_2) A set m is called ''infinite'' iff there are (proper) chains of elements of m.

The concept of a chain goes back to Richard Dedekind's proposal to define the set of natural numbers as equivalence classes of such chains. A chain can best be imagined as an infinite series of different elements. The definition is not circular, however, for the definition of chain does not invoke the infinity predicate so defined.

A third way of defining ''infinity'' makes explicit use of the natural numbers:

> (InfSet_3) A set m is called ''infinite'' iff there is a 1–1 mapping from the set of natural numbers into m.

This definition is, in a sense, equivalent to the aforementioned one.[8] But it makes clearer how a general concept of infinity can be based on the ''special infinity'' of the natural numbers. An infinite set is one that contains at least n elements—for *every* natural number n.

These are the most significant ways of talking about ''infinity'' in mathematics. There are, however, other contexts in which the word ''infinite'' is used in slightly different ways. For example, we call an interval of the reals ''infinite'' not for its being a set with infinitely many elements (every proper interval of the reals would be infinite in this sense), but for its being extended ''to the infinite in (at least) one direction.''

> (InfInt) A real interval I is called ''infinite'' iff it is unlimited in at least one direction, i.e., if there is no real number r^+ such that $I<r^+$ or there is no real number r_- such that $r_-<I$—that is, there is no upper or no lower limit to the numbers in the interval.

Sometimes, however, one calls an interval, which is finite in this sense, ''infinite'' with respect to a certain metric. Such a metric makes steps which are the same size according to the standard Euclidean metric of the reals have different sizes depending on their distance from the interval border. In such circumstances it is important to mind the difference between the senses in which something is called ''infinite.''[9] In mathematics, the sentence ''A finite interval is infinite'' does not necessarily give rise to suspicions of paradox. It is just a sentence with two different predicates designated with the same word ''infinite.''

There are also non-genuine ways of using the word ''infinity'' in mathematics—for example, when we say that the function f with $f(x)=x^2$ grows ''to infinity'' (when x becomes infinite). By this we mean that the function grows: $\forall x,y \geq 0\ (x < y \rightarrow f(x) < f(y))$, and that it grows unboundedly—that is, that there is no upper limit for the function values: $\neg\exists b \forall x\ (f(x) < b)$. We do not mean that the function has an infinite value. This is just a *façon de parler*, a non-literal piece of speech.[10]

In foundational discussions, we sometimes claim that a mathematical theory or an axiom system T makes infinity assumptions. There are several candidates for making such a way of talking precise:

- All models of T are infinite.
- For all natural numbers n, T proves ''there are at least $\underline{n}$ objects.''
- T proves ''there are infinitely many objects.''
- T proves ''there is an infinite object.''

These concepts are usually made precise by appealing to one of the set theoretic definitions of infinity. So, for example, a model of T is called infinite if its domain is an infinite set.

Infinity in Christian theology

In traditional Christian theology, only God is called ''infinite'' without further qualification. What is meant by the statement ''God is infinite''? A detailed analysis shows that ''infinity'' is used with respect to God in different senses.

In his *Summa Theologiae*, St. Thomas Aquinas develops his doctrine of God in the *quaestiones* following his famous five ways of proving the existence of God (STh I,2,3). Having considered God's simplicity, perfection and goodness, Aquinas treats God's infinity (STh I,7,1). His starting point is John Damascene's statement that God is ''infinite and eternal and uncircumscribable.''[11] Aquinas carefully distinguishes his understanding of infinity from the classical Greek notion. He points out that the natural philosophers before Aristotle had been right in calling the first principle ''infinite'' in that it generates more and more beings ''*ad infinitum*.'' But they had, Aquinas says, a wrong understanding of the first principle's infinity as they attributed infinity to matter (*materia*) and not to form (*forma*). Aquinas calls something ''infinite'' ''in virtue of the fact that it is not limited.'' According to the Aristotelian hylomorphism, form is limited by matter and matter by form. So there are two ways in which something can be free of limits—namely, in that it is pure form or in that it is pure matter. The infinity of matter means pure matter without form (*materia prima*), which is something imperfect. (*Materia prima* is, by the way, an abstraction and is not said to exist.) The infinity of form, on the other hand, means pure form, and this is a perfection. Aquinas recalls from his earlier arguments that ''the most formal of all things is *esse* itself.'' But God is his own subsistent *esse* (*ipsum esse subsistens*). Hence, Aquinas finally arrives at the conclusion that God is infinite. To put it in a nutshell: For Aquinas, God's essential infinity means being pure form without any limitation imposed by matter.

In a later article (STh I,7,3), Aquinas distinguishes sharply between essential infinity (as discussed above) and mathematical or quantitative infinity. He is well aware of the differences between them. And he is clear that quantitative infinity in the sense discussed in Aristotle's *Physics* is attributed to matter and therefore not to God.

Quantitative infinity is, however, not necessarily tied to the material part of reality. Since St. Augustine's time, quantitative infinity has had a definite place in theology, for example, in doctrines of God's knowledge. Against Stoic philosophy, Augustine teaches that God can comprehend what is infinite. His argument runs as follows: It is indisputable that there are infinitely many (finite) numbers. As God is omniscient, he knows them all. Hence, he knows infinitely many things. As St. Augustine puts it: ''Who, no matter how demented, would say that God's knowledge extends to a certain sum total of numbers but is ignorant of the rest?''[12] However, like most philosophers before and after him, St. Augustine held that while there are infinitely many numbers, there cannot be infinite numbers. It was not until Georg Cantor that philosophy was able to get rid of this philosophical prejudice.[13]

The First Vatican Council codified God's infinity into the official doctrine of the Catholic Church:

> The holy, catholic, apostolic and Roman church believes and acknowledges that there is one true and living God, creator and lord of heaven and earth, almighty, eternal, immeasurable, incomprehensible, infinite in will, understanding and every perfection.[14]

Here ''infinite'' is not used as a predicate proper but as something like a predicate modifier or a second-order predicate. In scholastic logic, such a usage of a word was called ''syncategorematic,'' for the word comes along with (=*syn*) a predicate (=*category*). It is not God who is called infinite, but his will, understanding, and other perfections. That these factors are infinite is probably meant to say that they do not suffer from limitations. With respect to his understanding, that is to say that God knows everything knowable, while we finite human beings only know a little part of what one can know.

In contemporary systematic theology, ''infinity'' is often used without clear indication of its meaning, although it is considered methodically central. The German theologian Wolfhart Pannenberg takes infinity even as a criterion for the adequacy of all theological propositions about God:

> Whatever contradicts the thought of the infinity of God cannot be part of a rationally defensible conception of the one God.[15]

Pannenberg's concept of infinity is hard to analyze. He derives it largely from Hegel, following him in a two-step process of thought: first, we perceive everything as finite in the sense of being *something*—that is, being determined. The negation of finiteness leads to the first concept of infinity. This infinite is, however, still limited by the finite as it is determined as ''the other'' of finitude. Therefore, this is only, in a sense, finite infinity. Second, then, one arrives at the ''true'' concept of infinity by negating even this contraposition of the finite and its (finite) negation in favour of a concept that somehow overarches both. This ''overarching'' concept of infinity can be identified, says Pannenberg, with the idealist concept of the Absolute for it means the complete absence of boundaries, limits or conditions of being. It is identical to Kant's concept of an *ens a se* (''*allgenugsames Wesen*'').[16] As to what concerns the criteriological function of this concept of infinity, Pannenberg relies on John Duns Scotus' argument that Divine infinity is a necessary requirement in order to show that there is only one God.

In general, ''infinity'' is assigned to God in several different senses. One may classify these senses in the following way:

- Quantitative sense:
  - God's properties are infinite in the sense that they are infinitely extended (e.g. omniscience = knowing infinitely many propositions);
  - God's properties are infinite in the sense that they are infinitely intense (e.g. omnipotence = having infinite power).
- Eminent sense:
  - God's properties surpass the properties of (finite) creatures or they are attributed to him in a way different from the way in which they are attributed to creatures (e.g. perfect goodness = being good in a way that somehow exceeds all creaturely goodness).
- Metaphysical/precategorical sense:
  - God's essence is beyond natural reality which is taken to be characterized by a metaphysical sense of finitude.

The quantitative sense shows up in traditional theology also as a predicate for non-divine entities—for example, in the debate on whether the world may have existed ''from eternity'' (in the sense of having not had a beginning before a finite amount of time), or in discussions about whether it was possible for God to create infinitely many souls, etc.

In contemporary theology, infinity is rather seldomly used in the quantitative sense; quite frequently it is used in the eminent sense; and most often it is used in the ''metaphysical'' sense. That ''infinity'' is used in these ways does not, however, mean that the sense of ''infinity'' would be explained accordingly. Almost always, the expressions ''infinite,'' ''infinity,'' ''finite,'' ''finitude,'' etc. are used as if everyone understood them without explanation. Sometimes circular explanations of their meaning are given or it comes without any explanation at all. Often it remains unclear what it means to say that the world, our mind, our lives, all of our abilities, plans, wishes, mental states, etc. are finite while God is infinite. A careful analysis of the uses of ''infinity'' in theological reasoning should bring more light to the meaning of these fundamental theological statements.

Cantor's ''absolute infinity''

Some people think that Georg Cantor, a mathematician with deep theological inclinations, had a clear conception of how mathematics and theology hang together. He used the expression ''absolute infinity'' in mathematical as well as theological contexts. So, is it not true that Cantor saw a deep connection here? I think it is. But I also think that the connection is much harder to see than many people think who invoke Cantor as authority.[17] As I have elaborated on this elsewhere, I want to sum up my position rather sketchily here.[18]

It is true that Cantor casually called both God and the ''size'' of proper classes ''absolutely infinite.'' The best-known pseudo quotation of Cantor's on these matters is that the transfinite cardinal numbers were ''steps to the throne of God.'' It is unsure whether Cantor really ever said so. But even if he had, this would not mean that he identified absolute infinity in the sense of set theory with the absolute infinity of God. Cantor's remarks are, however, not completely unambiguous at this point.

The best-known real quotations of Cantor are from his endnotes to §4 of the *Grundlagen*. They read:

> The absolute can only be acknowledged [[*anerkannt werden*]] but never known [[*erkannt werden*]][19]

and

> the absolutely infinite sequence of numbers thus seems to me to be an appropriate symbol of the absolute.

The exact sense of these passages is hard to grasp. Taking their contexts into account, one may arrive at the following reading.

Cantor calls God as well as proper classes ''absolutely infinite'' without in any way identifying them. They differ in the formal epistemological property that proper classes can be *approximately* known (i.e. large subclasses can be sets and therefore subject to set theoretical investigation), while God can never be known but only acknowledged. What God and proper classes have in common, however, is the fact both of them can *not* be *completely* known. This makes *On* a good symbol of the absolute infinity of God. At least, it is a better symbol than the sequence of finite natural numbers which was traditionally used as a symbol for the divine in theology and metaphysics. Now, Cantor says, there is a better candidate, for the natural numbers are still limited—namely, by the smallest transfinite ordinal number ω. So they do not really display the complete absence of limits in God. The class *On* is not limited in the same way—that is, by another number. And therefore, Cantor's main thesis is in my eyes a relative one: *On* or *Card* are more appropriate a symbol of the absolute than the series of natural numbers.

In addition to this symbolic relation, one can also find a parallel in methods between Cantor's mathematical theory and traditional theology. It can be put to the short slogan: no careless transfer of insights from the finite domain into the infinite one. As one must not simply transfer insights about creatures to God, so one must not simply transfer insights about finite numbers to infinite numbers (this, Cantor said, was the cardinal error of all proofs of the impossibility of infinite numbers).

In the form ''no transfer from the finite to the infinite,'' such a maxim played a major role in the traditional theological doctrine of God. In speaking about God, we cannot help but use the vocabulary of our everyday language that has gotten its meanings by being used in the contexts of our everyday lives. For such predicates, singular terms, relations, etc. are inappropriate for God, they have to run through a ''purgatorial process'' of *via positiva*, *via negativa*, and *via eminentiae*. To put it shortly: their positive content must be kept, the negative content of limitations, of the creaturely mode of being, must be crossed out, and their meanings must be ''boosted'' from the limited world of creature to the unlimited and uncreated creator. This ''purgatorial process'' precludes any simple transfer of propositions or meanings from the realm of the finite to the realm of the infinite. Such a preclusion holds for the realm of the infinite God as well as for the realm of infinite numbers.

Cantor sometimes explicitly crossed a bridge between mathematics and theology: depriving mathematics of actually infinite numbers means, in his eyes, to lose a cognition of God. He explains this thesis with respect to a passage from the First Vatican Council, and a verse from the Bible. The First Vatican Council taught that God is ''inexpressibly loftier than anything besides himself which either exists or can be imagined.''[20] With respect to the original Latin formulation ''*concipi possunt*'' which was translated as ''can be imagined,'' Cantor adds that the loftiness of God would be the more considerable the more extended the area of things below him is; and the more conceptual resources there are (e.g. Cantor's mathematical theory), the more we can guess how far extended it really is. The biblical Book of Wisdom says that God has ''disposed all things by measure and number and weight'' (Wisdom 11:20). The more numbers there are, Cantor argues, the more things God can dispose—that is, the higher is his power. Cantor wrote about this in more general terms to the Dominican father Thomas Esser:

> Every extension of our insight into what is possible in creation leads necessarily to an extended cognition of God.''[21]

Common aspects of mathematical and theological concepts of infinity

We have seen that ''infinity'' is used in mathematics as well as in theology in several different ways. Some of these ways are clearly related; in other cases it is not clear whether such relations hold.

If, in theology, ''infinity'' is used in a quantitative sense, then it is clearly related to mathematics. The mathematical definition tells us exactly what it means that, say, the set of all truths God is said to know is infinite. And the same holds for the other examples of quantitative infinity in theology: infinite amounts of time, infinitely many living beings, infinitely many states of affairs God is said to be able to bring about, etc. Theology, then, can profit from the clear definitions and the interesting results of mathematical research in the realm of infinity. Just to note one example: if a theological argument presupposes that a certain class of things cannot be enlarged for its being infinite, theology can recognize that this argument is unconvincing. The theologian can try repairing it by finding other reasons that the class cannot be enlarged or by finding a weaker consequence of the class's infinity which may suffice for what follows in the argument.

In many cases, however, the application of quantitative concepts of infinity in theology seems not very appealing—for example, when divine omniscience is taken to mean that God's knowledge is infinite in that it comprises infinitely many truths. This is not the meaning of the statement that God is omniscient. Its meaning is that God knows everything there is to know (the Latin *omnes/omnis/omnia* means everything, all, complete). Hence, omniscience may *imply* infinite knowledge (e.g. with the help of the further premise that there are infinitely many truths), but it does *not mean* it. Just consider the (counterfactual) case that there were only finitely many things to know in our world. Then a God who knew them all would rightly be called ''omniscient,'' but he would not have infinite knowledge.[22]

As to what concerns the eminent way of infinity talk in traditional theology, it is not clear what ''infinity'' exactly means in such contexts—for example, in the context of the traditional doctrine of ''degrees of perfection.'' In order to make this way of talking more precise one may be tempted to think about applying quantitative concepts. But this would lead to further problems. If God is said to be infinitely good, how is that to be spelled out? Does it presuppose sort of a ''measurability'' of goodness? What, then, is an infinite degree of goodness? What about the fact that many measures with ''infinite'' degrees can be mapped onto a finite measure on the reals? If God's infinite goodness is just to mean that his goodness surpasses all the goodness found in creatures, such surpassing could be realized also by a large, but finite amount.

With respect to the third way of using ''infinity'' in theology, the—as I call it—''metaphysical'' sense, the trickiest question is what its precise sense is. In discussing Pannenberg's approach, I have indicated what such a ''metaphysical sense'' could be, but neither do I think this is completely perspicuous, nor do I think the Hegelian concept is promising. It does not even cover the mainstream metaphysical uses of ''infinity'' in systematic theology. (One can read this off the careless use of ''finitude.'' If finitude was just the opposite of Hegelian infinity, finitude would be a heavily complex concept.)

It is an interesting question whether there is a common property (or, better, in the Fregean sense a *Merkmal*) of the several different infinity conceptions in theology and mathematics. We touched upon the most promising candidate when scrutinizing Aquinas' argument for God's infinity. He says:

> Note, therefore, that something is called infinite in virtue of the fact that it is not limited.[23]

Note that in the Latin original, the corresponding expressions are ''*infinitum*'' (infinite) and ''*finitum*'' (limited). This explanation is quite word-oriented or—as some people call it—etymological. One may have the impression that it is very broad, general, and poor in content. But the advantage of this explication is that it seems to cover most other kinds of infinity. Taking ''having no limits'' as the most general concept of infinity, one has to note, however, that infinity is seldom used to describe something as having no limits whatsoever (*infinitum simpliciter*), but rather is used to mean having no limits in a certain respect or of a certain kind (*infinitum secundum quid*).[24] If things are quantitatively infinite, they are infinite *secundum quid*, with respect to the category of quantity, but not *simpliciter*.

Endnotes

1 An earlier version of this article was presented during the conference ''Numinous Numbers,'' October 2009, in Boston, USA. The author can be contacted by email at: ph-th@rub.de. I am indebted to Jim Bradley for helpful comments.

2 For a classification of the different kinds of relations, see Ivor Grattan-Guinness, ''Christianity and Mathematics: Kinds of Links, and the Rare Occurrences after 1750,'' *Physis*, new series 37:2 (2000): 467–500, reprinted in Ivor Grattan-Guinness, *Routes of Learning: Highways, Pathways, and Byways in the History of Mathematics* (Baltimore: Johns Hopkins University Press, 2009), 288–322.

3 Loren Graham, ''The Power of Names,'' *Theology and Science* 9:1(2011): 157–164.

4 See Christian Tapp, *Kardinalität und Kardinäle* (Stuttgart: Franz-Steiner-Verlag, 2005).

5 H. Weyl, *Philosophie der Mathematik und der Naturwissenschaft*, 7th edition (Munich: Oldenbourg, 2000), 89.

6 As Ludwig Neidhart suggests by the title of his voluminous monograph, *Unendlichkeit im Schnittpunkt von Mathematik und Theologie* (Göttingen: Cuvillier, 2005).

7 This is true at least as long as one does not ''localize'' mathematical truths like others in the mind of God. This would amount to a pantheistic position. Note that this is not the same as saying that God knows all truths or the like, which would not necessarily amount to pantheism.

8 Equivalence in the sense of mutual implication is dependent on the background theory. I do not want to elaborate on this in the context of this article.

9 The same holds, *mutatis mutandis*, for Einstein's famous aphorism that space is infinite but limited. ''Infinity'' is used then in the sense of the metric, while the limitation stems from the embedding in Euclidean space. (Confusion may arise if one considers the Euclidean metric thus induced into the embedded space. According to this metric, the infinite space is finite.)

10 I think that this is what Carl Friedrich Gauß had in mind when he made his famous claim that talking about infinity in mathematics is nothing but a ''*façon de parler*''; see his letter to Schumacher (12 July 1831), in *Carl Friedrich Gauss—H.C. Schumacher Briefwechsel*, part 1 (= Carl Friedrich Gauss: Werke, Ergänzungsreihe, vol. 5, part 1), ed. C.A.F. Peters (Altona, 1860, reprinted Hildesheim: Olms, 1975), 268–271.
11 Quotations are taken from the English translation of the *Summa Theologiae* by Alfred Freddoso; http://www.nd.edu/~afreddos/summa-translation/TOC.htm.
12 Augustine: *De civitate Dei* XII,19=*The City of God Against the Pagans*, trans. Philip Levine (Cambridge, MA: Harvard University Press, 1966), vol. IV, 89–91.
13 See Tapp, *Kardinalität und Kardinäle*, especially 75–116.
14 Norman Tanner (ed.), *Decrees of the Ecumenical Councils*, vol. 2: Trent to Vatican II (London: Sheeed & Ward, 1990), 805.
15 Wolfhart Pannenberg, *Metaphysik und Gottesgedanke* (Göttingen: Vandenhoeck & Ruprecht, 1988), 28, my translation.
16 Pannenberg, *Metaphysik und Gottesgedanke*, 21, 28–33.
17 For reliable general information on Cantor and his ideas on infinity, still see Joseph Dauben, *Georg Cantor, His Mathematics and Philosophy of the Infinite* (Cambridge, MA: Harvard University Press, 1979). Cantor's works are edited as Georg Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed. E. Zermelo (Berlin: Springer, 1932).
18 See Christian Tapp, ''Absolute Infinity – A Bridge between Mathematics and Theology?'', *Foundational Adventures: Essays in Honor of Harvey M. Friedman*, ed. Neil Tennant (London: College Publications, forthcoming).
19 ''*Werden*'' added to the German quotation.
20 Tanner, *Decrees of the Ecumenical Councils*, 805.
21 See letter (CanEss96) in Tapp, *Kardinalität und Kardinäle*, 308, and the related commentary on 86 (my translation).
22 This argument depends on the premise that knowing a finite number of propositions does not imply that one knows infinitely many propositions—for example, all the logical consequences of the propositions.
23 Thomas Aquinas, *Summa theologiae*, p.I q.7 a.1 co.
24 See *ibid.*, p.I q.7 a.2 co.

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