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On Some Philosophical Aspects of the Background to Georg Cantor's theory of sets*

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Résumé : Georg Cantor a cherché à assurer les fondements de sa théorie des ensembles. Cet article présente les différentiations cantoriennes concernant la notion d'infinité et une perspective historique de l'émergence de sa notion d'ensemble.

Abstract: Georg Cantor sought secure foundations for his set theory. This article presents an account of Cantor's differentiations concerning the notion of infinity and a tentative historical parsepctive on his notion of set.

*I am indebted to Joseph W. Dauben for his helpful advise. — Comments to the author are welcome.

1 Introduction

Historical accounts of the life and work of Georg Cantor (1845-1918) have generally focused chiefly on his mathematical creations, i. e. on these parts of his scientific production which are — valued by lines of today's mathematical judgement — important for the development of set theory and mathematical logic. However, the founder of set theory also sought secure foundations for his emerging new theory. His interests in the philosophical (and even theological) background of set theory were indeed quite broad, and although he tried to defend his new mathematical theory against attacks he anticipated from philosophers and theologians, the importance he ascribed to these activities is often underappreciated.

This paper aims at providing a fresh and new account of two main aspects of Cantor's philosophy of mathematics (if it may be referred as such), namely to the notion of sets (and the question of their reality), and to the notion of infinity. If it is correct that Cantor's main mathematical achievement was the invention of the theory of infinite sets and numbers — transfinite set theory — then the interrelation between the notion of set and the notion of infinity is of utmost importance for the investigation of their (intended) philosophical foundations.

What this paper offers is more a brief and introductory discussion of Cantor's philosophy of mathematics, rather than a comprehensive and all-embracing consideration.

2 Philosophical Apologetics Regarding the Notion of Infinity

The well-known and for the most part accurate biography by Walter Purkert and Hans Joachim Ilgauds[1] considers Cantor's philosophy in the course of a chapter devoted to his personality. Within this chapter, his philosophical efforts come up between some remarks concerning music, holidays, and strange religious concerns for the older Cantor as reflected in his edition "Ex Oriente Lux".[2] This gives the impression that philosophy was an element of Cantor's private life, or even a side-occupation for him like playing the violin, having little to do with his scientific work. In contrast, Joseph W. Dauben's biography of Cantor[3]

[1] [Purkert & Ilgauds 1987].
[2] [Cantor 1905].
[3] [Dauben 1979].

gives more attention to Cantor's philosophical acitivities and takes them more seriously.

But Cantor himself appreciated philosophical studies as being of great importance for the foundations of mathematics in general, and for his own scientific work in particular.[4] As early as in his "Historische Notizen über die Wahrscheinlichkeitsrechnung"[5] of 1873, Cantor attached particular importance to philosophical investigations of mathematics. With reference to probability theory, for example, he argued that the demand of mathematics for truth and validity[6] calls for a philosophical critique and a suitable philosophical reason or even "metaphysics" (as — following Cantor — the French would call it).

Set theory's inventor was convinced of the need to defend his new theory of infinite numbers against various objectors and as far as I can see, this conviction was due not only to Cantor's scientific views of the necessity of philosophical scrutiny, but also to a desire to pave the way for the acceptance of his theory and to lower the opposition to and prejudices against it. Why were such resistances to await? — At first, the general view concerning the infinite and its place in science in general and in mathematics in particular was quite different from ours today. We could refer for example to the polemical position of Kronecker, for whom only the natural, but not even the rational numbers could naturally be said to exist. But such reference to one single position cannot really answer the aforementioned question.

Instead, the key point is the overall historical situation of mathematics at that time. Shortly before Cantor's main ideas began to emerge, the works of Cauchy, Weierstraß and Dedekind had shown a way out of the disagreements stemming from careless use of the notion of the infinitely small. The invention of a precise notion of limit with its — as we would say today — typical nesting of quantifiers permitted the elimination of a very problematic notion of infinity, one with which mathematicians had struggled for a long period of time. And so we see the general situation at the time Cantor introduced his new theory: Mathematics had only recently been freed from the problems of one notion of infinity, and now

[4] Lacking a general term in English for what is called "Wissenschaft" in German, "science"/"scientific" are used here in a broader sense comprising not only the natural sciences, but also disciplines like philosophy, mathematics, theology, and so on.
[5] [Cantor 1873]. — This is an *early* date (Cantor was about 28 years old) for his philosophical activities have usually been attributed to a later period of his life.
[6] More precisely stated, Cantor spoke of the basic concepts and foundations of mathematics, which take on a certain real validity ("Begriffe und Grundlagen, die [...] eine gewisse reale Gültigkeit in Anspruch nehmen") *in view of the applicability of their results.* See [Cantor 1932, 365].

Cantor appears, introducing a new one, and in fact a very much stronger one, because the new notion could be contrasted with the improper, weak old forms!

Cantor took up a main distinction from these discussions: he distinguished between the proper and the improper infinite. The development of his terminology concerning this point is schematically presented in the following table:

1883[7]	"Uneigentlich-Unendliches" (improper infinite) "unbegrenzt Wachsendes" (something growing without limit) "synkategorematisches Unendliches" (synkategorematical infinite)	"Eigentlich-Unendliches" (proper infinite) "Vollendet-Unendliches" (completed infinite)
1886[8]	"potentiales Unendliches" (potential infinite) "Indefinitum"	"Aktual-Unendliches" "aktuales Unendliches" (actually infinite)
1887[9]	"ἄπειρον" (something boundless)	"ἀφορισμένον" (sth. definite / marked off)

Both notions (the improper and the proper infinite) have in common that they indicate a magnitude that exceeds every finite magnitude, but the meaning of "exceeds" differs. In modern terms, in the case of the potential infinite, a process or a functional relation is needed by means of which the magnitude can be made larger than any given finite value, but, strictly speaking, the value of the magnitude remains finite. In contrast, the actually infinite magnitude *is* larger than every finite magnitude, independently of any kind of process or function.

Cantor dealt with the improper infinite in two different ways. On the one hand, he took the underlying problems to be resolved by the works of Cauchy and Weierstraß. The mathematical notion of limit, which is crucial for the buildup of analysis, was made precise by the definition that a is the limit of a sequence a_n if and only if for every ε there is a number n_0 such that for all $n > n_0$ the (absolute) difference between a and a_n is less than ε — a definition of limit in which the notion of

[7] See [Cantor 1883] = [Cantor 1932, 165-6,175,180] and more often.
[8] See [Cantor 1886a] = [Cantor 1932, 372-3] and more often.
[9] See [Cantor 1887] = [Cantor 1932, 401] and more often.

infinity no longer occurs.[10] On the other hand, Cantor was somewhat attracted by an argument of Constantin Gutberlet to the effect that a concept of the potential infinite requires an underlying actual infinite just as (by analogy!) the possibility of moving forward without limit on a path requires the actual infinity of the path.[11]

Dealing with the second, the proper or actual infinite was what Cantor wanted to defend from criticism. As for Cantor, philosophy was not excused from the demand of searching for the truth, and thus he believed that the history of philosophy must be heard and that former positions must be taken into account. Thus Cantor took the objections seriously that many philosophers (and also some mathematicians) had made against the theoretical treatment of the actual infinite. He wanted no arguments to remain undiscussed, no counter positions to stay undisproven so that the extent of his philosophical task of defending philosophically a theory of real infinity ranged from Aristotle through Augustine, Thomas Aquinas, Spinoza, Leibniz, and Kant up to contemporary schools and positions including neoscolastics and neokantians.

In my opinion, a detailed examination of all these positions and Cantors counter-arguments would be of considerable interest, especially from two perpectives, namely first: Does Cantor's presentation of the arguments against the actual infinite do justice to their originators' intentions? And second: Does his rebuttal succeed? Unfortunately, these questions are beyond the scope of this article.

Instead, I want to propose a very brief overview summarizing the most important reasons advanced against the actual infinite, and then offer an equally brief account of Cantor's counter arguments.

Primarily, there are basically 4 assumptions used to argue against the real existence of the actual infinite:

(1) There are only finite numbers. (Aristotle)

(2) Something finite would be destroyed if there really were something infinite. (Aristotle)

[10] With respect to this point, David Hilbert and the intuitionists shared a common counter-position. They dealt primarily with the question of whether or not the quantifiers ($\forall\varepsilon$, $\exists n_0$ and $\forall n > n_0$) presume completed totalities or not. Cantor may not have been opposed to this line of thought, but see his second approach following. — For Cantor's first approach, see for example [Cantor 1932, 92-3]. Compare as well Cantor's corresponding treatment of the real numbers.
[11] See for example [Gutberlet 1878, 11-5], and Cantor's references to Gutberlet [Cantor 1932, 394].

(3) The human mind is finite.

(4) The non-finite or the absolute is not capable of determination / negation (Spinoza and Leibniz, among others).

Let me roughly describe Cantor's lines of critique.

In Cantor's view, Aristotle concludes thesis (1) via a *petitio principii*: He was only aquainted with counting finite sets and inferred that there are only finite numbers and *therefore* only finite sets. But thereby he equated what he is acquainted with to what there is. This thesis cannot really support the assertion that there cannot be infinite numbers, and it fails completely as a result of the existence of transfinite number theory.[12]

The bare pronouncement (2) cannot claim to be valid, as Cantor argues with reference to ordinal arithmetic, which demonstrates the fact that the destruction of something finite by something infinite does not always occur, but only in the case of adding the infinite ω to 1 from the right, as opposed to adding it from the left:

$$\omega + 1 \neq \omega = 1 + \omega$$

(or the other way around, depending on the definition of ordinal addition).[13]

With considerable insight Cantor detected a vicious circle regarding arguments that start at the finiteness of the human mind. Most authors — as he notes — conceived the finiteness of the human mind in terms of its restriction to a capacity for forming finite numbers. But then this thesis presupposes what it wants to prove: the impossibility of forming infinite numbers.[14]

We have still one argument left and in the case of (4), we have to make the following distinction: On the one hand, Cantor accepted the classical theorem "omnis determinatio est negatio" (every determination is / includes a negation) and agreed to the second part of thesis (4), namely that the absolute is not capable of any determination or negation, especially not of a determination by numbers. But on the other hand, he strongly opposed the first part of this thesis, the rash identification of the non-finite and absolute.[15] This is the systematic germ cell of Cantor's later polished distinctions between various notions of infinity

[12]See especially [Cantor 1932, 174].
[13]See [Cantor 1932, 174-5].
[14]See [Cantor 1932, 176-7].
[15]See [Cantor 1932, 175-6].

that he eventually worked out in detail. Up to now he had only stressed the difference between the potential, improper infinite (a boundlessly growing magnitude) and the actual, proper, real infinite.

But now it became a matter of differences for Cantor within the real infinite itself — about which he speaks of "modifications". Without such differences, the actual infinite qua negation of the finite could well be identified with the absolute, highest, indeterminable infinite — which was not seldom equated to God. This rang a warning bell of the guardians of faith who suspected Cantor was advocating a pantheistic metaphysics[16] when he asserted later on that the actual infinite existed not only in the human mind, but also in the created real world. It was to counter such suspicions that Cantor introduced a further conceptual distinction between the transfinite and the absolute as two kinds of actual infinity. He described three ways in which the actual infinite could occur: as the absolute, the transfinite, and transfinite numbers. He characterised these in the following way:

	absolute	transfinite	transfinite numbers
1885	*in Deo extramundano aeterno omnipotenti sive natura naturante*	*in concreto seu in natura naturata*	transfinite, actual-infinite ordertypes
1887	in the highest perfection, in the absolutely independent, extramundane being, *in Deo* realized	represented in the dependent, created world	conceived by thinking *in abstracto* as mathematical magnitude, number or ordertype
short	*in Deo*	*in concreto*	*in abstracto*

Due to these distinctions the absolute no longer had to be considered as the only antipode of the finite, and Cantor's mathematical theory of actual infinity was freed from the suspicion of pantheism. He was also able thereby to theoretically define theology as the science which investigates what can be said about the absolute by human beings. Therefore Cantor drew the scholastic conceptual distinction between "*natura natu-*

[16]Cantor himself once called Hegel a pantheist, see [Cantor 1886a] = [Cantor 1932, 376].

rans" and "*natura naturata*", which goes back at least to Johannes Scotus Eriugena (died 877). In his main work, "*De divisione naturae*", which was strongly influenced by ideas from Augustine and Boethius, he developed the first complete medieval system of metaphysics, starting with a division of all beings according to the act of creation. Therefore, he discerns the following four kinds of entities:

creans non-creatum	God as creator
creans creatum	ideas (?)
non-creans creatum	spatio-temporal beings
non-creans non-creatum	God as destination of beings

This conceptual system developed through the era of medieval scolastics and was adopted by Giordano Bruno (1548-1600) in the 16^{th} and Baruch de Spinoza (1632-1677) in the 17^{th} century (in the verbalization "natura naturans") and remained popular with 19^{th}-century neoscolastics. It may well have come to Cantor via Spinoza and the neoscolastics.

Turning back to Cantor's threefold distinction of the actual infinite, the relevant difference between Cantor's viewpoint and that of 17^{th}-century philosophy depends upon another conceptual distinction. Aside from the absolute infinite and the finite there is a region of infinity between them: the region of the transfinite. Transfinite objects — such as transfinite numbers, for example — really belong to the sphere of the actual infinite. They do not *become* infinite, they do not grow without finite limit, but actually are definite magnitudes of infinity, so that they belong to the actual, not to the potential infinite.

The transfinite also differs from the absolute, but in what ways? — Correspondence between Cantor and the Jesuit Cardinal Johannes B. Franzelin from 1886 shows that both — Cantor and Franzelin — agreed that the transfinite has to be conceived of as something still enlargable.[17] So like the finite it is still enlargable, but the difference with respect to the finite is that before as well as after any possible enlargement, the transfinite was and stays actually infinite. On the opposite, the absolute was taken to be "wesentlich Unvermehrbares", something that essentially could not be enlarged, or better: something that was in principle not capable of enlargement. Making this distinction, Cantor was able to respect the — so to say — holy and untouchable area of God, although he was dealing with definite actually infinite entities. He succeeded in destroying the Cardinal's suspicions of pantheism-like metaphysics, and

[17]See Franzelin's letter to Cantor of the 26.1.1886 [Cantor 1932, 385], referring to Cantors paper "Zum Problem des actualen Unendlichen" [Cantor 1886b].

thereby made room for a further and fruitful discussion between Cantor and Roman-Catholic theologians in the last fifteen years of the 19^{th} century.

Despite this very brief account of Cantor's "philosophical coordinate system", and in spite of leaving many details out of accout, it seems clear that for him philosophy was by no means a minor concern or peripheral occupation.

3 Development of the Concept of "Set"

This section is devoted to the development of Cantor's notion of set and to philosophical questions about its meaning.

Arguably, the most famous formulation defining the notion of set is from Cantor's last mathematical publication, the two-part "Beiträge zur Begründung der transfiniten Mengenlehre"[18]:

> Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen. [By a "set", I conceive any union M of definite, well discriminated objects m of our imagination or our thought (which are called the "elements" of M) into a whole.][19]

This definition gives rise to two lines of considerations: first, because it was widely misunderstood in the context of the subsequent foundational crisis, and second, because it marks the final stage in the development of Cantor's notion of set.

3.1 Cantor's Definition and the Paradoxes of Set Theory

The aforementioned definition has often been misunderstood as comprising something like an unrestricted comprehension principle:[20] If we take (but only for the sake of the argument) "Zusammenfassung" (embracing) to mean "formally definable" (by a formula ϕ), then *for every formula* ϕ

[18][Cantor 1895] and [Cantor 1897].

[19][Cantor 1895] = [Cantor 1932, 282], transl. from [Dauben 1979, 170]. — For the sake of simplicity most *loci citati* are given this way: when citing original papers, page numbers are given according to the reprinted edition of in Cantor's collected works. Translations without attribution are the author's.

[20]Concerning Cantor and the paradoxes of set theory, consider [Purkert 1986].

(of any language whatsoever) there must be a set M_ϕ consisting of all objects satifying ϕ, i. e.: $M_\phi = \{x|\phi(x)\}$. Therefore, it has been said that this definition leads to the set theoretical paradoxes (Russell and Burali-Forti, for example).

But this interpretation is surely incorrect. When Cantor required the "*Zusammenfassung zu einem Ganzen*" (*union into a whole*), this was exactly the *caveat* he made against "taking-as-a-whole" elements that *cannot* be taken to form a whole without contradiction. Some further evidence is provided by two letters that Cantor wrote to David Hilbert, one on 26 September, 1897, and the other on 2 October, 1897. In the second, Cantor emphasized the inconsistency of a "set of all Alephs", noting that such a thing might be well-defined but not "*fertig* definiert" (*completely* defined) because:

> Ich sage von einer Menge, daß sie als fertig gedacht werden kann, [...], wenn es ohne Widerspruch möglich ist [...], alle ihre Elemente als zusammenseiend, die Menge selbst daher als ein zusammengesetztes Ding für sich zu denken. [I say about a set that it can be thought of as complete, [...], if it is possible without contradiction [...], to think of all of its elements being together and of the set as a composed thing in its own.][21]

And he continues referring explicitly to his "Beiträge":

> Darum definire ich auch im ersten Artikel der Arbeit Beiträge zur Begründung der transfiniten Mengenlehre gleich im Anfang die "Menge" [...] als eine "Zusammenfassung". Eine Zusammenfassung ist aber nur möglich, wenn ein "Zusammensein" möglich [d. h. widerspruchsfrei denkbar, C. T.] ist. [In the first paragraph of the work Beiträge zur Begründung der transfiniten Mengenlehre, just at its beginning I therefore defined the "set" as a union. But a union is possible only, if a being together is possible.][22]

Twelve years earlier, in his "Grundlagen einer allgemeinen Mannigfaltigkeitslehre" of 1883, Cantor wrote:

> Unter einer "Mannigfaltigkeit" oder einer "Menge" verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken läßt, d. h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann. [By an "aggregate" or "set" I mean generally any multitude which can be thought of as a whole, i. e., any collection of definite elements which can be united by a law into a whole.][23]

[21] [Purkert & Ilgauds 1987, 226].
[22] [Purkert & Ilgauds 1987, 227].
[23] [Cantor 1883] = [Cantor 1932, 204].

Leaving aside the observation that "durch ein Gesetz" (by a law) sounds as if Cantor adhered to the widespread but problematic identification of sets and definable sets (which in the case of a countable language would make the number of different sets be countable), we have to conclude at this point that Cantor's definition of "set" does not lead to antinomies, because it simply does *not* permit an *unrestricted* comprehension principle.

3.2 Cantor's Definition as a Final Point of Development

It is surely correct when Joseph Dauben states that Cantor's famous 1895 definition served to free him "from the particular character of point sets and to produce a completely general theory".[24] But we can go even further and point out that such generality was (to a certain degree) present right from the beginning, and that the formulation was not only a liberating but also a restrictive one at the same time.

First we should note that in his early writings, Cantor used the words "Menge" (set), "Mannichfaltigkeit"/"Mannigfaltigkeit" (manifold), "Inbegriff" (collection) and even "System" (system) as basically synonymous.[25] Later on (probably after his detection of the contradiction in the notion of the set of all cardinals), Cantor distinguished two types of "Vielheiten" (multiplicities): the consistent ones which can be conceived without contradiction as forming a whole, i. e. whose elements can be thought of as being present altogether at the same time, and the inconsistent ones which cannot be so conceived. From then on, Cantor reserved the term "set" for consistent multiplicities only.[26]

In the 6-part series of articles "Über unendliche lineare Punktmannigfaltigkeiten" (1879-1884), we find some textual evidence for a remarkably

[24] [Dauben 1979, 170].
[25] [Cantor 1874] used the word "Inbegriff" for sets. (As opposed to Zermelo [Cantor 1932, 118] I would not say that Cantor used the expression "Gesamtheit" as a synonym for "Inbegriff". When Cantor said "Die reellen algebraischen Zahlen bilden in ihrer Gesamtheit einen Inbegriff von Zahlgrößen" [Cantor 1932, 115], then he used "bilden in ihrer Gesamtheit" — and not "bilden eine Gesamtheit" — to underline the shift of talk from single algebraic numbers to the set of algebraic numbers and to make clear that his use of "Inbegriff" does not refer to the individual numbers but to their comprehension into a set.) [Cantor 1878] uses "Inbegriff" and "Mannigfaltigkeit" synonymously, see esp. [Cantor 1932, 120]. [Cantor 1879] states explicitly that "Menge" (set) is something like a shorthand for "Mannigfaltigkeit" (manifold), see [Cantor 1932, 139].
[26] See his letter of 26 September, 1897, to Hilbert [Purkert & Ilgauds 1987, 224-6] or his letter of 28 July, 1899, to Dedekind [Cantor 1932, 443-7].

wide notion of set. In N° 3 (1882) Cantor wrote:

> Auch der Mächtigkeitbegriff [...] ist so wenig auf die linearen Punktmengen beschränkt, daß er vielmehr als Attribut einer jeglichen wohldefinierten Mannigfaltigkeit betrachtet werden kann, *welche begriffliche Beschaffenheit ihre Elemente auch haben mögen.* [Even the notion of cardinality [...] is not restricted to linear point sets, so that it can be regarded as an attribute of any well-defined manifold, *whatever the conceptual nature of its elements may be.*][27]

This quotation is taken from a passage which clearly argues for the extension from concepts introduced for the investigation of linear point sets to their use in the context of non-linear point sets, which is still a genuinely mathematical context. But it is striking that for three paragraphs no mention is made of this mathematical context. Admittedly, some incertitude, the passage could cautiously be interpreted as reflecting a conception of set much broader than considered normally. Perhaps it is a conceptualization meant to include also sets of non-mathematical objects or even objects of the natural world.

This interpretation of Cantor's earlier position is also supported by the quotation from the "Grundlagen" of 1883 given above (p. 166). There, Cantor considered generally *any* multitude that may be taken as a whole. Later, in a note, Cantor emphasized the presumed closeness of his notions to Plato's ideas,[28] which also contributes to a sense of generality.

Whether the last point of this interpretation be admitted or not, it disappears or clearly plays no role in the 1895 definition, where only objects of our imagination or thought are allowed as elements of sets. This surely excludes objects of the natural world, and so at the end of the century, for Cantor the notion of a set *and its elements* seems to have been completely restricted to the area of thought and imagination, meaning that they are purely mental objects.

3.3 The Ontology of Sets

Prominent lines of thought in the philosophy of mathematics investigate questions like "Is there a real universe of sets?" "Where can it be found?" "Does the notion of set refer to objects with a certain reality?" "Is set

[27] [Cantor 1882] = [Cantor 1932, 150].
[28] See [Cantor 1932, 204].

theoretic platonism true?" and so on. Strange to say, but these endeavours go along with some kind of (postmodern?) capitulation in the face of the question of truth. Different frameworks are considered, classified and analysed, but which framework to choose for oneself is no longer a question of truth but a matter of taste.

Perhaps, it would be more fruitful to change the point of view to a different question "How did mathematics and its philosophy arrive at this problem of seeking a realization or instantiation of its main concepts?" It seems to me that it was a by-product of axiomatization and a strictly formal standpoint that resulted in the notion of set losing contact with its philosophical origins, a rest of which we can find in Cantor's viewpoints. So much for something like a philosophical confession which seems to me necessary in order to explain why I do not consider in detail so many standard and common sense views on this subjects in what follows.

It seems to me to be of great interest for the philosophy of mathematics to know the ontological status that Cantor assigned to his sets. In the foregoing subsection we have already seen some evidence for the hypothesis that Cantor's early notion of set was a broad one, possibly embracing sets of physical objects. Therefore, I think it going too far if one ascribes mind-world dualism and the subordinated location of sets in the realm of mind-made constructions having something like a "realization" in the real world to Cantor's philosophical standpoint.

Indeed, Cantor understood the whole numbers and the ordertypes:

> als Universalien, die sich auf Mengen beziehen [...] wenn von der Beschaffenheit der Elemente abstrahiert wird. [as universals, which are related to sets [...] if one abstracts from the nature of its elements.][29]

Similarly, cardinal numbers emerge from ordertypes by a further abstraction (which Cantor denoted with a second bar over the set symbol). If ordertypes and numbers are obtained from the real world by an act of abstraction, then it does not seem suitable to speak of dualism, at least not in the strict sense of the word which refers to the doctrine claiming that the world consists of two different substances. What Cantor wanted to convey is that ordertypes are certain set universals, and the difficulty with the common classification of Cantor's views as "platonistic" is that he in fact did not really resume the discussion of realism of universals, debated in philosophy since the Middle Ages.

But there are also problems with a broader understanding of "dualism", because Cantor — despite his general criticism and rejection

[29] [Cantor 1887] = [Cantor 1932, 379].

of Kantian philosophy — shared the conviction with Kant (and with Thomas Aquinas, among others), that apriori knowledge also presupposes at least *some* experience. For Cantor it was quite clear that even mathematical concepts presuppose experience in general, a conviction — by the way — which Frege misleadingly criticized as Cantor's "psychologism".

In section 2 above we have already considered a further distinction Cantor made in the notion of the transfinite. He discerns on the one hand the transfinite occuring in nature or, if we want to be more precise and do not mind running the risk of sounding like a theologian, we should say: the transfinite occuring in *created* nature, and on the other hand the transfinite occuring as order types or numbers respectively. To put it yet another way: He distinguishes the transfinite *in concreto* and the transfinite *in abstracto.*

This difference is — on first view — far removed from our understanding of set theory as an abstract mathematical science just from its beginning. (A view, which I take to be responsible for persistent questioning philosophically of the applicability of mathematics in the natural sciences.[30]) This is all closely connected with the notion of set that Cantor held during the second half of the 1880s. At the beginning, Cantor needed to emphasize certain ontological elements as part of the background for set theory, which seem to have vanished from his later publications.

When Cantor defines ordertypes as universals related to sets which arise from them by abstraction, these elements can be — as Cantor says — "beliebige wohlunterschiedene Dinge" (arbitrary well-discerned things). In my opinion, to conceive of these things as elements of a set, seems *not* as yet to be a form of abstraction from *their* concrete nature.

We can find some evidence for this view some pages later in the "Mitteilungen" where Cantor describes ordertypes as "das intellektuale Abbild einer uns gegenüberstehenden Menge" (the intellectual image of a set that exists apart from us).[31] It is not implausible to conclude from this reference to "*intellectual* image" that the entity which is imaged need not itself be an intellectual entity. Moreover, Cantor considers it as a characteristic of the difference between sets and ordertypes that the elements of the set must be grasped as "getrennt" (discrete) whereas the elements of the ordertype are "zu einem Organismus vereinigt" (united into an organism).

[30] Recall Cantor's early statement that probability theory also needed a philosophical critique with respect to its applicability; see footnote 6.

[31] See [Cantor 1932, 380].

Further evidence for this conception of sets as consisting of — so to speak — natural elements can be found in Cantor's 1883 address to the meeting of the "Vereinigung deutscher Naturforscher und Ärzte" in Freiburg, when he stated that one of the most important problems of set theory, which he believed to have solved, consists in the challenge of determining the various magnitudes or powers of sets *present in nature* via the notion of an ordinal number.[32]

I think that we must therefore conclude that Cantor's original approach allowed (at least the elements of) sets to be worldly entities, to be sets of arbitrary natural things such as tables, chairs, pencils, bottles or beer mugs. The first proper act of abstraction enters when the ordertype and the cardinal number are abstracted from the set under consideration. To obtain the cardinal number of the set it is necessary to forget about the specific properties of its elements and their order. They must be understood — as Cantor says — as units. By abstraction, each element becomes a "one" so that the set as a whole becomes a set of ones or of units forming a oneness.

A minimal interpretation simply argues that an eventually given ordering of the set's elements has to be disregarded. But why then speak of an "organic growth together into each other" to make such an assertion? I think that Cantor's emphasis on the organic and uniform character of cardinal numbers can be understood properly only if we consider his intention to defend cardinal numbers as real, actual *numbers*, as falling under the notion of number, where the "old" notion of number clearly must be broadened, but keeping its essential features. And therefore Cantor accepted the classical Euclidean postulate regarding numbers as real unities, in Greek: μόνας, in which a multiplicity of units must be uniformly connected.

Another interesting conclusion can be drawn from Cantor's review of Frege's "Grundlagen der Arithmetik" of 1884. In this review, Cantor defines the power or cardinality of a given set to be the general notion under which all sets fall that are equivalent to the given set. (By the way, we should remark that this concept together with the hypothesis that elements of sets need not be abstract objects resembles Russell's reconstruction of natural numbers as classes of tupels of worldly objects of the right length, or perhaps, the other way round.)

By now it is apparent that Cantor's widely-known and famous definition of "set" in his "Beiträge zur Begründung der transfiniten Men-

[32] Published in the "Mitteilungen" as a part of a letter to Kurd Laßwitz. See also [Dauben 1979, 291].

genlehre" of 1895 and 1897 was not something like a constant factor throughout his development of set theory, but was — at least in its mature and more exact version — a subject to historical changes, and — perhaps also — to periods of hiatus.

4 Conclusion

What happened to Cantor's notion of set during the decade from 1886 to 1895? What discoveries or arguments prompted him to modify his conceptualization and treatment of "sets" in particular? There is some evidence that Frege's penetrating criticism of what he called Cantor's psychologism may have had an influence on this point. But some researchers, as Dauben, argue for a more cautious assessment of the Cantor-Frege connection, for as he notes that "there are nevertheless changes in the *Beiträge* with which Frege would have agreed."[33] Was it an advance in thought alone, or conceptual progress that Cantor made in conjunction with philosophical or even theological influences? — The answers to these questions are a matter for further research.

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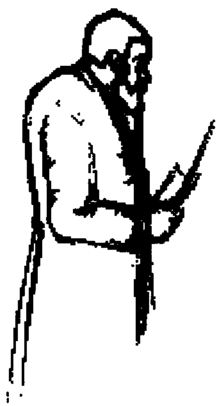

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