CHAPTER 8

Bolzano's Concept of Divine Infinity

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Cuius regni non erit finis.
His kingdom will have no end.
—Nicene-Constantinopolitan Creed

Infinity is central to each of the three areas in which Bernard Bolzano (1781–1848) had expertise: mathematics, philosophy, and theology. The Bohemian priest dealt with all the respective concepts in these disciplines: quantitative infinity in mathematics, qualitative infinity in philosophy and theology, and some of their mutual interrelationships.

I focus on "infinite" used as a divine predicate. Since this use of "infinite" is related to a quantitative use in several ways, we will also dwell on these.

QUANTITATIVE AND DIVINE INFINITY IN THE *PARADOXIEN DES UNENDLICHEN*

In his late major work *Paradoxien des Unendlichen* (*PU*) (in English, *Paradoxes of the Infinite*),[1] Bolzano set out to resolve the most significant mathematical paradoxes of infinity. For him, infinity was a key concept

for mathematics, physics, and metaphysics (*PU*, §1). Hence, he found it crucial to dissolve all kinds of paradoxical "*Schein*" (appearance) and to make it clear that the concept of infinity is perfectly rational.

To do so, Bolzano first presents a definition of infinite quantities: "I propose the name infinite multitude for one so constituted that every single finite multitude represents only a part of it" (*PU*, §9).

At first glance, this definition seems defective. It immediately raises two questions: What is a finite multitude? How could every finite multitude be a part of all infinite multitudes?

To begin with the second question: take, for example, the finite multitude that consists of me and Bolzano. Compare this two-element set with the set of natural numbers. To be sure, the two-element set of me and Bolzano is not part of the infinite set of natural numbers. So how could the set of natural numbers fulfill the definition of infinity, according to which every single finite multitude *must somehow be a part of* an infinite set? This puzzle is quite easily solved by interpreting more precisely the definition of an infinite multitude as a multitude of which "every single finite multitude represents only a part." If we take "represent" to mean that every finite multitude can be *mapped* onto a part of any infinite multitude, and not that every finite multitude *is* a part of it (in the sense of being a subset), the second problem is solved.

With respect to the first question (What is a *finite* multitude?), we should note that Bolzano proposes his definition at the end of a lengthier paragraph in which he talks about a series of individuals of a species *A*. He says that such a series may contain more or less numerous sets of terms:

> In particular, the series may contain so many terms that it cannot, compatibly with taking in and exhausting all the individuals of that species, be conceded to have a *last term*; a point which we shall handle in greater detail in the sequel. Assuming it for the present, I propose the name infinite multitude. (*PU*, §9; emphasis in original)

Hence, Bolzano introduced his incomplete definition of an infinite multitude in the context of a discussion about a series without a last term. Not having a last term, however, is a sufficient, but not necessary condition for being infinite. (Standard Cantorian set theory gives us lots of examples for its being nonnecessary. Take, for instance, the second infinite ordinal number ω + 1. It is infinite but has, according to its standard representation, a last element, namely ω.)

Hence, I propose the following for making sense of Bolzano's definition of quantitative infinity:

1. A series is called "certainly infinite" iff it has no last term.
2. A series is called "finite" iff it can be mapped one-one onto a part of every certainly infinite series.
3. A series is called "infinite" iff every finite series "represents a part of it," that is, can be mapped one-one onto a part of it.

In this manner, we have captured the substance of Bolzano's definition and used two of his own suggestions to make the definition fully acceptable: the concept of a series without a last term, and the idea of representations as one-one mappings.

Today, it has become mathematically standard to define infinite sets as sets that allow a one-one mapping onto a proper subset of themselves. Astonishingly, Bolzano already noticed that infinite sets have this property and even that this property is characteristic of infinite sets (*PU*, §20). However, it was Richard Dedekind (1831–1916) who made this into a definition of infinity (Dedekind 1888, note on no. 64). In Dedekind's work, a variant of Bolzano's definition also survived in the form of the alternative definition that a set *S* can be called "infinite" if for every natural number *n*, *S* has at least *n* elements.[2]

So much for Bolzano's definition of quantitative infinity. In *Paradoxes of the Infinite*, he is quite successful in employing this concept to resolve all kinds of alleged paradoxes traditionally assumed to be generated by the concept of infinity.

But some philosophers claim that, besides this relatively precise notion, we also need another concept of infinity: a concept of *qualitative* infinity. Hegel, for instance, famously claimed that quantitative infinity is "bad infinity" and that we have to form a "dialectical" notion of "true infinity" (*wahrhafte Unendlichkeit*) that transcends the opposition of the finite and the infinite, in which the infinite was merely the negation of the finite (see Hegel [1830] 1970, §§93–95). Bolzano distinguished carefully between different Hegelian claims: (1) Hegel is completely right, says Bolzano, in criticizing the concept of potential infin-

ity as "bad" infinity. What people call "potential infinity" is a property of growth processes, not of quantities. As the values of potentially infinite growth processes may always stay finite, potential infinity is a crude concept in Bolzano's eyes (*PU*, §11): "Infinite" shall be used as a predicate of quantities,[3] but with respect to quantities it does not make sense to call them infinite when they in fact always stay finite (§12). (2) However, Bolzano claims, Hegel is completely wrong to identify mathematical infinity with potential infinity. A limitlessly growing magnitude may always stay finite. In contrast, all the substantial mathematical uses of infinity are those of actual infinity. So whatever Hegel is talking about, it is not mathematical infinity. (3) Bolzano is very skeptical about an additional qualitative concept of infinity. However, he does not bluntly reject it as incoherent or empty. He just thinks it superfluous for the following reason: "What I refuse to admit is only this: that the philosopher knows any object to which he is entitled to attach the predicate of infinitude without having first established that in some respect or other that object exhibits infinite quantity, or at least infinite multitude" (§11).

What, then, would be the best touchstone for this claim? To prove it in the case of God!

> Now if I can once show that in God Himself, the Being Whom we regard as the most perfectly one, aspects can be found under which we see even in Him an infinite multitude, and if I can show that we attribute infinitude to Him under those aspects alone, then it will scarcely be necessary to go on and show that similar considerations lie at the bottom of all the other cases where the idea of the infinite holds good. (*PU*, §11)

So, God is a kind of test case for Bolzano: if, even in the case of God, his thesis about quantitative infinity being basic is true, then there is a very good reason to take it as being generally true. The same holds for the Hegelian doctrine of the All: if the All is all there is, such that outside of it there is nothing, then this All comprises an infinite multitude of entities, because it is an aggregate of all existing things, even nonactual things, such as absolute propositions and truths. Hence, even in the case of the All, it is unnecessary to deviate from Bolzano's account

of quantitative infinity as primordial infinity. To coin a phrase: "No infinity without quantitative infinity!"

From the point of view of a philosopher of religion, this is a surprising thesis, since there are very few, if any, philosophical accounts of divine infinity that make use of quantitative concepts. But this is Bolzano's view of divine infinity, which we shall explore in this chapter.

Is Bolzano's thesis true? Is it the case that even in God we find quantitatively infinite aspects? Bolzano says yes:

> I say then: We call God infinite because we are compelled to admit in Him more than one kind of force possessing an infinite magnitude. Thus, we must attribute to Him a power of knowledge which is true omniscience, and which therefore comprises an infinite set of truths, to wit, all truths and so forth. (*PU*, §11)

This is a short statement of Bolzano's doctrine of divine infinity. We find it in a much more developed form in an earlier major work, the *Religionswissenschaft*.

DIVINE INFINITY IN THE *RELIGIONSWISSENSCHAFT*

The title of Bolzano's *Religionswissenschaft* (*RW*) (in English, *Science of Religion*) might be misleading. It is not a book of religious studies, but one about philosophical and revealed theology. Bolzano treats the divine attributes twice in this book: first in the context of natural theology (part I), then again in the context of revealed theology (part III). The two treatments are mutually consistent, though the revealed theology account is enriched by considerations about the "moral" and "real" value of each element of the doctrine and about its historical roots (which are, for the most part, biblical).

The Divine Attributes in Natural and Revealed Theology

When Bolzano treats the divine attributes in the context of natural theology, he scarcely mentions infinity.[4] This is completely different in revealed theology where he explicitly uses infinity as the central conceptual tool for explicating God's omni-perfection, and "infinity" even

shows up in several section headings.[5] Bolzano's idea was that infinity is central for the theoretical development of the divine attributes. Even if he does not mention the word "infinity" in the first part, he uses it when he looks back to the first part in considering the "necessity of a revelation even for the most erudite people" (*RW*, I, §99). There he summarizes his earlier treatment of the divine attributes, saying that natural theology leads to the insight that a divine being exists and that it represents the epitome of all perfections insofar as it possesses "infinite power, wisdom, and holiness."[6] Natural theology, however, cannot reach much farther in envisioning the divine. In spelling out the reasons for this claim, Bolzano assigns infinity a crucial role:

> The first deficiency of this [the natural] religion reveals itself, if we want to envision the aforementioned properties of God somewhat more perspicuously. Because all divine powers and qualities (*Beschaffenheiten*) have a certain infinity, while our finite understanding struggles in conceiving the infinite, a humble person will become the more timid in her judgments about God, the more carefully she considers that it is the infinite being about which she dares to judge. (*RW*, I, §99)[7]

Hence infinity, even though not treated explicitly, was a key concept for Bolzano's natural theology.

Natural Theology (*RW*, part I)

Bolzano develops his natural theology based on the concept of God as an "unconditioned real" (*unbedingt Wirkliches*; *RW,* I, §66). This means: something is God if it is real and is not dependent upon anything. Using this notion, Bolzano presents a proof of God's existence (§67) and elaborates some of His properties: necessary existence (§69), substantiality (§70), the unconditioned nature of His powers (§71), immutability (§72), uniqueness (§73), and, finally, omni-perfection (§74).

Bolzano then considers the different powers of God (*RW*, I, §75). "Power" means "the potential to effect something." Effects can be internal, in the case of immanent powers, or external. Bolzano distinguishes four immanent powers—to think, to feel, to wish, and to will—and two external powers—to create, and to change. All these powers, which we know in ourselves, we also meet in God, except the power to wish, which would make no sense to ascribe to a being enjoying highest beatitude (§75, no. 6; see also §77, no. 1). The guiding principle for Bolzano's considerations is that God must have these perfections in the highest degree possible in themselves and in conjunction with the other perfections.

Let us now consider two highly instructive examples of Bolzano's treatment of divine properties, or powers: the power of knowledge (omniscience) and the external powers (omnipotence).

*God's Omniscience (*RW*, I, §76)*

An omni-perfect being must have the power to think, which must be as great as it can be, given its other powers. Because the power to think is not limited by other powers (instead, the other powers are limited by the power to think), it must be as great as it can be in itself: as the power to reach all the goals of thinking, to know all truths *an sich*. There are, however, infinitely many truths *an sich* (as Bolzano putatively proves elsewhere; see my next main section). Hence, God's power to know is infinite in the sense that it extends over infinitely many truths *an sich*. Note, however, that Bolzano does not explicitly talk about the infinity of God's power to know here, but that omniscience is conceived of as knowing *all* truths *an sich*. That these are infinitely many is not a *conceptual part of omniscience* as it is conceived in the first part of *Religionswissenschaft*.

God's Omnipotence

God's omnipotence is God's power to produce external effects (*RW*, I, §79). In us, such a power is variously limited by, for example, limits of our power to think, as we forget things or wrongly think them to be impossible for us, thus inhibiting the producing of certain external effects. In God, however, "all these limitations cease to apply" (§79, no. 4). Hence, Bolzano concludes, God's power to produce external effects extends to *all* that (1) is possible in itself, (2) is compatible with the moral law, and (3) requires a determining ground of its being.

Note, again, that Bolzano does not speak of infinity in this context. There is not even the obvious implication of an infinite set, like the set of all truths *an sich* in the case of omniscience. Interestingly, all this is completely different when Bolzano considers the divine attributes in the context of revealed theology, in the third part of *Religionswissenschaft*.

Revealed Theology (*RW*, part III)

In part III of *Religionswissenschaft*, where Bolzano develops a "catholic dogmatics," he explicitly addresses infinity when reconsidering the divine attributes treated in part I.

Bolzano begins by considering God's omni-perfection (*RW,* III, §43). All possible perfections are united in the highest possible degree in God's essence. This teaching of Christianity is so obvious that many philosophers have taken "all-perfect being" to be the definition of God. (Bolzano's definition is, instead, that God is an unconditioned real, as we have seen in *RW*, I.) However precise our concepts are, God remains incomprehensible to us in the sense that all teachings and theoretical insights about Him provide no account of how He is in Himself, but only a picture or a shadow image (III, §47). Bolzano says the "true reason" for God's incomprehensibility is that God has "totally *infinite powers.*" For example: His power of knowing extends to all truths. Since there are *infinitely many truths*, to understand God completely one would have to understand *infinitely many truths,* and this is impossible for finite beings such as human beings. Hence for us, He is incomprehensible.

Bolzano calls God's power of understanding "God's infinite intellect" (*RW*, III, §52) and repeats the claim of the first part that God's power to think extends to whatever is knowable, that is, to all truths *an sich* (§52).[8] But this is puzzling. The quick transition from *an infinite capacity to know* to *knowing infinitely many truths* is not completely obvious. Having an infinite capacity to know (*RW*, III) and knowing everything knowable (*RW,* I and III) are conceptually distinct. So how can the transition be justified? According to Bolzano, there are infinitely many truths *an sich* to be known. (We will consider this in the next section.) That assumption is true. But it is not part of the concept of "knowing everything" that the extension of "everything" is an infinity of truths. Furthermore, the converse implication does not hold: from the fact that someone knows infinitely many truths, one cannot validly conclude that she knows everything. Imagine I know all true equations of natural numbers (0 = 0, 1 = 1, 2 = 2, etc.). Then I know infinitely many truths, but not all, by far.[9] We will come back to this puzzle.

On Bolzano's understanding, "God knows everything" (*RW*, III, §55) contributes much to the venerability of God. To illustrate what it means to "know everything," Bolzano digresses from his method of systematic argument: God knows all numbers, all combinations of numbers in sums of two, three, or four, and so on, in products and quotients. He knows the many different lines in geometry, the different angles, triangles, quadrangles, and so on, the orders of curved lines, the many different surfaces, bodies, spaces; the mechanical combinations of many different physical powers. In the empirical world, He knows the many stars, planets, comets; how many living and dead organic and inorganic beings there are on planet earth, how many atoms in each one of the thousands of mosquitoes flittering in the evening sun; and the relations between any two atoms in the whole universe (which God must arrange in the correct way in order to stabilize the cosmos as it is). If we consider all this, then, Bolzano says, "we must sink on our knees before the infinite, because we feel, what it means to be infinite" (§55). God's infinity is what requires us to worship and adore Him.

We have here a complete reversal of Heidegger's much later critical remark that one cannot sink on one's knees and pray, and sing, or dance, in front of a *causa sui*, which stands *exemplum pro specie* for strictly philosophical concepts of God (see Heidegger 1957, 70). For Bolzano, in contrast, it is precisely divine infinity which illustrates that the being which fulfills the formal concept of an unconditioned real is also worthy of the highest adoration and worship.

THE INFINITY OF TRUTHS

In our analysis of Bolzano's doctrine of divine infinity, we have been puzzled by the logical connection between knowing everything knowable and knowing infinitely much. Part of a solution consists in Bolzano's claim that the class of truths *an sich* is infinite. He offers proofs of this, for example, in the *Wissenschaftslehre* (*WL*) (in English, *Theory of Science*) (*WL*, I, p. 146, 147; *RW,* I, p. 35; *PU,* §13).

These proofs proceed inductively. According to the principle of complete induction, they have two parts: first, that there is at least one truth, and, second, that for every natural number n, if there are at least n truths, then there are at least $n + 1$ truths. By the definition above, it follows that the class of all truths is infinite.

There is at least one truth. This is proven by an indirect proof (*RW*, I, p. 35; *WL,* I, p. 145). Assume that it is not the case that there is a true proposition.[10] But, "[It is not the case that there is a true proposition]" is a proposition.[11] Hence it cannot be true, because we have assumed there is no true proposition. Therefore, by semantic descent, it is not the case that it is not the case that there is a true proposition, that is, our assumption was false and there is a true proposition.

The induction step from *n* true propositions to *n* + 1 is more interesting, but its idea is almost the same. I reproduce Edgar Morscher's semi-formal reconstruction (Morscher 2011, 202):

> Assume the Inductive Hypothesis (IH): there are at least *n* true propositions, say $S_1, \ldots, S_n$. Then we have to prove: there are at least *n* + 1 true propositions. We do this by the following indirect proof, reducing the negation of our claim to the absurd:

1.	There are not at least *n* + 1 true propositions.	(assumption)
2.	There are no more than *n* true propositions.	(from 1 by PL)
3.	There is no true proposition except $S_1, \ldots, S_n$.	(from 2 and IH)
4.	[There is no true proposition except $S_1, \ldots, S_n$] is a proposition.	(premise)
5.	[There is no true proposition except $S_1, \ldots, S_n$] is true.	(from 3 by semantic ascent)
6.	[There is no true proposition except $S_1, \ldots, S_n$] is a true proposition.	(from 4 and 5)
7.	[There is no true proposition except $S_1, \ldots, S_n$] is different from each proposition of the $S_1, \ldots, S_n$.	(premise)
8.	There is a true proposition different from $S_1, \ldots, S_n$.	(from 6, 7; contradiction to 3)
9.	There are at least *n* + 1 true propositions.	(from 1, 3, 8 by indirect proof)

Although this proof is logically basic, it adduces two premises that require consideration. The first premise is that [There is no true proposition except $S_1, \ldots, S_n$] is a proposition (line 4). This is not to say much more than that "There are not more English sentences than $S_1, \ldots, S_n$" is

an English sentence. The only special feature is Bolzano's distinction between linguistic sentences and sentences *an sich*. Linguistic sentences express sentences *an sich* if they fulfill certain semantic requirements (like being unambiguous; see Morscher 2011) that are fulfilled in the case of our first premise. The second premise is that the "new" proposition [There is no true proposition except $S_1, \ldots, S_n$] is distinct from each of the propositions $S_1, \ldots, S_n$ (line 7). Bolzano argues for this premise, for example, in *PU*, §13:

> For if we fix our attention upon any truth taken at random . . . and label it A, we find that the proposition conveyed by the words "A is true" is distinct from the proposition A itself, since it has the complete "proposition A" for its own subject.

The main objection to Bolzano's proof, and similar proofs (from Dedekind, Cantor, and others) is Frege's assertion that "p is true" does not express anything more than "p." But for his sentences *an sich*, Bolzano offers a clear-cut criterion of identity: propositions are identical if and only if they have the same subject-idea and the same predicate-idea. Therefore, "p is true" and "p" can be identical only if [p] is also the subject-idea of p and [is true] is also the predicate-idea of p, that is, only if p is the self-referential proposition [p is true]. In our case, in contrast, $S_1, \ldots, S_n$ are all part of the predicate-idea of p and we can quite safely assume that they are not part of the predicate-ideas of $S_1, \ldots, S_n$, respectively.

Given Bolzano's proofs that there exist infinitely many truths *an sich*, part of our problem is solved. It is now clear that a being that knows everything knowable, in fact knows infinitely many truths, namely, all the infinitely many truths *an sich*. Because the converse inference is invalid, however, there remains the problem of explaining how Bolzano could be justified in skipping so quickly from knowing infinitely much to knowing everything knowable.

A FORMAL BACKGROUND CONCEPT OF INFINITY

My thesis is that, in fact, there is no way to get from knowing infinitely much to knowing everything knowable. Hence, in order to in-

terpret Bolzano in a coherent way, we need to ask whether our reconstruction of Bolzano's "infinite capacity of knowledge" as the capacity to know infinitely much is correct. This interpretation was suggested by Bolzano's (*PU*, §11) claim of the priority of quantitative infinity. Nevertheless it seems much more coherent to ascribe to Bolzano another, quite formal concept of infinity in the background. What concept is that?

The Formal Background Concept . . .

In *Wissenschaftslehre* (I, §87) Bolzano discusses how other philosophers have conceived infinity. He chooses four candidates and discusses them.

(1) The first candidate is *indeterminateness*: something is infinite insofar as it cannot be precisely specified or determined. Bolzano rejects this conception on the basis of several arguments. Let us consider one of them. Bolzano says that there is only one understanding of infinity as indeterminateness that makes sense; that infinite sets are not determinable by their proportion to finite sets. This is correct, says Bolzano, but indeterminateness in one respect does not mean determinateness per se. Hence one cannot say that infinite sets are indeterminate per se. He presents a knockdown counterexample: the line between two points in Euclidean space contains infinitely many points. The set of these points is therefore infinite, although it is precisely determined by the two terminal points. Hence it does not make sense to identify infinity and indeterminateness.

(2) The second candidate is *nonaugmentability*: something is infinite insofar as it cannot be augmented or increased. This definition is clearly false, says Bolzano, for one can increase the set of points on a line between two points (which is infinite) by simply extending the line beyond one of the terminal points.[12] Hence there are infinite sets that are clearly augmentable and, therefore, nonaugmentability cannot be a necessary condition for infinity.

(3) The third candidate is "*being larger than every possible set*." But this definition is unfulfillable, because every set that might fall under it must, by definition, be larger than itself, which is impossible.

(4) The fourth candidate is "*having no end, limit, or bound*." In Bolzano's eyes, this is the best attempt at defining "infinite." But it must be

corrected because X's being infinite does not mean that X has no limit in every respect. Again, he offers the counterexample of a line between two terminal points in Euclidean space. Having two terminal points, the line has limits in a certain respect.

Having no limits is the formal background concept I claim to be central for Bolzano's whole doctrine of infinity. Before I argue that it is indeed justified to ascribe this concept to Bolzano, even though he had some reservations concerning it, let us see how it fits into his thinking about infinity, mathematical and divine.

Bolzano's idea of mathematical infinity was inspired by the natural idea that counting (with constant speed, in finite time, and idealized beyond our creaturely constraints) will not come to an end. To count a set *s* means to present a part one-one mapping from the natural numbers onto *s*. Coming to an end in counting *s* means that all elements of *s* have been reached. A set *s* can then be called "finite" if all countings of *s* come to an end. A set *s* is infinite iff there is a one-one mapping of the natural numbers onto *s*. The connection between the precise mathematical definition of infinity and the general philosophical concept of "having no limits" is then as follows: a set is infinite (in the mathematical standard sense) iff it has no limit with respect to idealized counting.

Bolzano's distinction between "having no limits in a certain respect" and "having no limits in every respect" is helpful, especially when one considers mathematical theorizing about infinity after Bolzano. Cantorian transfinite ordinal numbers, for example, do not fulfill the criterion of "having no limits in every respect" because each one is limited by the next transfinite ordinal number. But they have no limits in a certain respect, namely, with respect to idealized counting of their members.

Is God infinite in the sense of having no limits whatsoever? On Bolzano's theory, we cannot say so without reservation. God's perfections are restricted to what is possible in itself, compatible with the moral law, and in need of a ground of its being. He therefore seems not unrestricted in every respect, unless such seeming restrictions do not really restrict divine power but rather exclude mere impossibilities from the extension of the powers ascribed to God in the form of the divine predicates.[13] Nevertheless, the concept of being without restrictions is helpful in developing

Christian doctrine. For example, if Christian faith calls God's intellect, wisdom, or will "infinite," it implies that certain restrictions, which we know from our human nature, do not apply in God's case.

. . . and Its Justification

Accepting this background concept of "having no limits" leads to terminological difficulties with the use of "being unlimited" and "being infinite" in today's mathematics. According to this use of language, there are infinite things that are limited. (For example, every transfinite ordinal number is an infinite number, but limited, for example, by the next transfinite ordinal number. Or, take Bolzano's set of points on a line that is infinite but limited.) Also, there are unlimited things which are finite (for example, the space between the curve $y = 1/x^2$ and the x-axis, taken from $x = 1$ to $+\infty$). But let us leave this aside here in favor of finding a coherent interpretation of Bolzano independently of the question of which terminology best expresses it.

I see two main reasons for parenthesizing Bolzano's reservations toward this concept and, instead, understanding him with the help of it. The first reason is that Bolzano's treatment of omnipotence strongly suggests this formal understanding. Bolzano asks how great our power to influence external things is. Our human powers over external things are already limited by the limitation of our other powers, such as our restricted cognitive capacities and our weak will. God, in contrast, does not suffer from such limitations. His will is perfect, and He can do whatever is possible and compatible with his other attributes (*RW*, III, §57). Hence, his external power is infinite in just the formal sense of being limitless.

The second reason in favor of my thesis is that the formal background conception helps to solve the second part of the puzzle of the quick transition from an infinite capacity of knowing to knowing everything knowable. The infinity of God's intellectual capacities does not *mean* that God knows infinitely many truths *an sich*, but that these capacities have no limit. An unlimited capacity for knowledge, however, includes the capacity to know everything knowable. For why should a truth that is knowable in principle be excluded from being known by a being with the unlimited capacity of knowledge?

CONCLUSIONS

Divine infinity is an integral part of Bolzano's concept of God. It is a test case for his broader claim that the concept of quantitative infinity is more basic than all other concepts of infinity; that all meaningful discourse about infinity presupposes quantitative infinity. This thesis from §11 of the *Paradoxes of the Infinite* must not be understood as reductivist, as though Bolzano would *only* accept quantitative infinity. He claims only that in each case in which we are entitled to use qualitative concepts of infinity, which are vaguer and more opaque, we must also be able to find aspects of the subject matter that exhibit the more precise quantitative infinity.

Ascribing to Bolzano a formal background conception of infinity as limitlessness or unrestrictedness, as suggested by his treatment of omnipotence, helps us to interpret his doctrine of God in a coherent way. However, what exactly it means to be unlimited, to suffer from no limitations in a certain respect, remains a topic for further philosophical analysis. Does every conception of "limits" require a metric or at least a total linear ordering? How can one align such a concept with the mathematical way of speaking, according to which there are unlimited finite magnitudes and also limited infinite magnitudes? Finally, what "respects" are to be considered, in the case of divine infinity, as unrestricted in certain, if not all, respects?

NOTES

An earlier version of this chapter was published in German (see Tapp 2011).

1. The standard English reference edition of *Paradoxien des Unendlichen* is Steele's edition (Bolzano 1950). For the German original text, there is a more recent edition (Bolzano 2012).

2. For Dedekind's famous definition of infinite chains, see Dedekind 1888 (71–73). Both definitions are equivalent over standard set theories.

3. Cf. the Aristotelian dictum the Scholastics used to phrase as *infinitum quantitatem sequitur*, "the infinite follows [= belongs to the category of] quantity" (Aquinas, *Summa contra Gentiles* 1. 1, c. 43).

4. Bolzano mentions infinity only in these contexts: the infinite regress in the causality proof for the existence of God that he critically analyzes in *RW*, I, §68; the description of the divine power of sensing (*Empfindungskraft*) as providing God with

beatitude of an infinitely high degree (*RW*, I, §77); the discussion of necessity to create infinitely many beatific creatures in the world, which is infinite with relation to space (§81, no. 2); and a cursory mention of God's infinite wisdom in the context of a discussion of the immortality of the soul (§85).

5. For example, *RW*, III, §52, has the heading "Die Lehre von Gottes unendlichem Verstande" ("The Doctrine of God's Infinite Understanding"), and "Die Lehre von Gottes unendlich vollkommenen Willen" ("The Doctrine of God's Infinitely Perfect Will") is part of the heading of *RW*, III, §57.

6. ". . . daß dieses Wesen als der Inbegriff aller Vollkommenheiten, und sonach begabt mit einer unendlichen Macht, Weisheit und Heiligkeit gedacht werden müsse" (*RW*, I, §99). This and the following English translations from the *Religionswissenschaft* are my own.

7. ". . . der erste Mangel in dieser Religion verräth sich, wenn wir uns die so eben erwähnten Eigenschaften Gottes noch etwas deutlicher vorstellen wollen. Weil nämlich alle göttlichen Kräfte und Beschaffenheiten eine gewisse Unendlichkeit haben, das Unendliche aber von unserem endlichen Verstände schwer aufgefaßt werden kann: so wird der bescheidene Mensch um desto schüchterner in seinen Urtheilen über Gott, je reiflicher er erwägt, daß es das unendliche Wesen sey, das er hier zu beurtheilen waget" (*RW*, I, §99).

8. God can even know our errors but, as Bolzano puts it, only "formally," not "materially." God can know that someone mistakenly believes the false proposition *p* but, in that case, God cannot know *p* itself. See also *RW*, I, §76.

9. *RW*, III, §53, offers a hint toward an explanation of this tension. There, Bolzano quotes Psalm 139 at some length to demonstrate that his doctrine is in accordance with scripture. The most important verses are 17–18a (KJV): "How precious also are Your thoughts to me, O God! How great is the sum of them! If I should count them, they would be more in number than the sand."

To be sure, the number of grains of sand is a speaking metaphor for an incredible large number. Verse 17b, however, calls their number "great," while Bolzano's version calls it "infinite" (my translation, see *RW*, III, §53). This close connection between infinity and greatness also helps to explain the fact that Bolzano calls God's intellectual powers "infinite."

10. To stay short and to facilitate understanding, I simply use "proposition" for Bolzano's "Satz an sich."

11. In the Bolzano literature "[p]" is used to denote the *Satz an sich* expressed by the linguistic sentence p.

12. Georg Cantor later made augmentability the main criterion of distinction between the transfinite and the absolute infinite. The transfinite shares with the finite the property of being augmentable, while the absolutely infinite is not augmentable. What exactly Cantor meant by this is a matter of discussion, however. See Tapp 2014.

13. Thomas Aquinas explicitly excludes inconsistent descriptions of states of affairs from the descriptions that are covered by his doctrine of divine omnipotence. Dealing with the question of whether this would not mean a restriction of divine power, he says that "whatever implies contradiction does not come within the scope of divine omnipotence, because it cannot have the aspect of possibility. Hence it is better to say that such things cannot be done, than that God cannot do them" (*Summa theologiae* I, q. 25, a. 3, c.a.).

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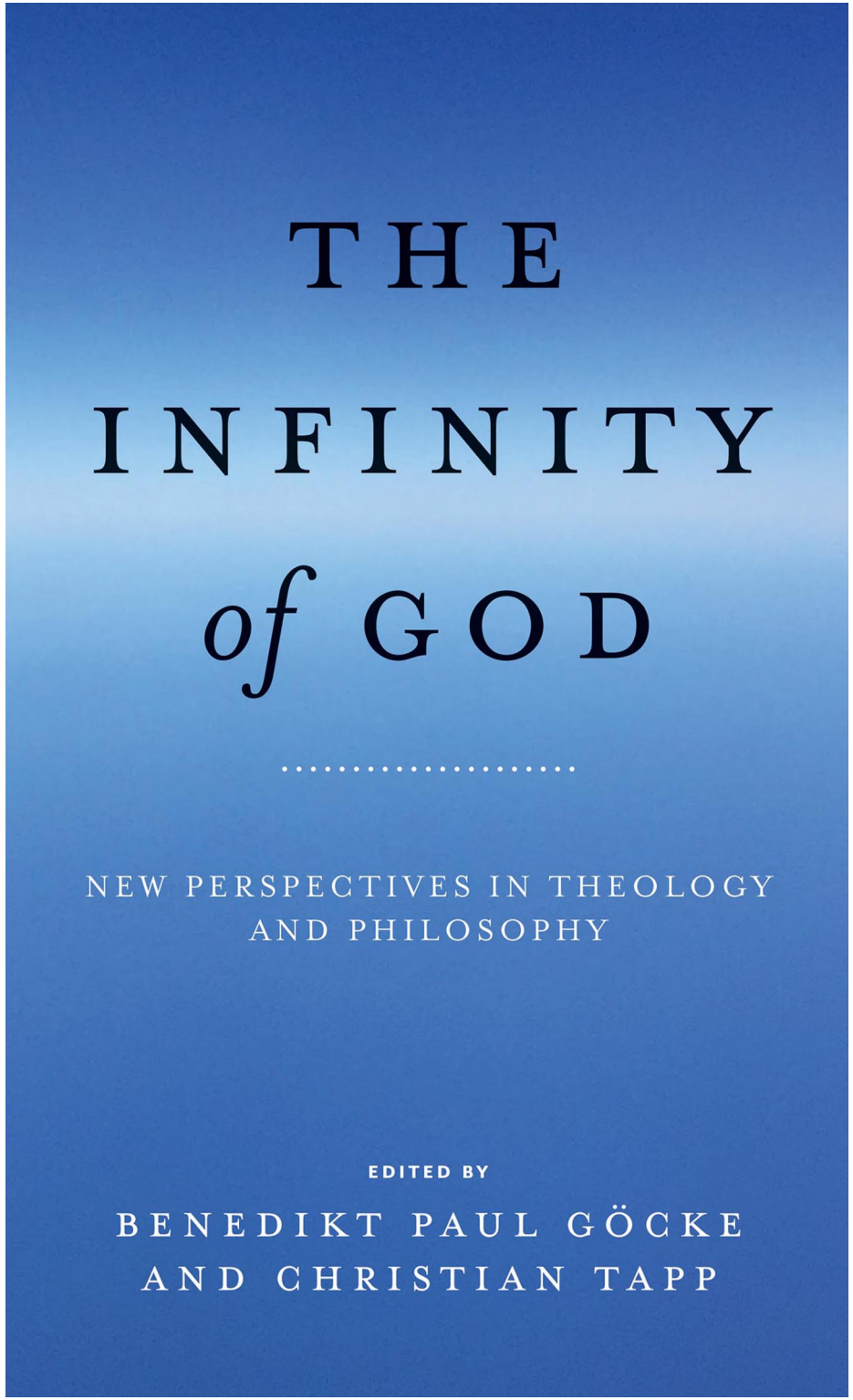
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