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The influence of Spinoza's concept of infinity on Cantor's set theory

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ABSTRACT

Georg Cantor, the founder of set theory, cared much about a philosophical foundation for his theory of infinite numbers. To that end, he studied intensively the works of Baruch de Spinoza. In the paper, we survey the influence of Spinozean thoughts onto Cantor's; we discuss Spinoza's philosophy of infinity, as it is contained in his *Ethics*; and we attempt to draw a parallel between Spinoza's and Cantor's ontologies. Our conclusion is that the study of Spinoza provides deepening insights into Cantor's philosophical theory, whilst Cantor can not be called a 'Spinozist' in any stricter sense of that word.

1. Introduction

Georg Cantor (1845–1918) was not only the founder of set theory; he also cared very much about the philosophical tasks connected with it. Right from the beginning of Cantor's university education, the works of Baruch de Spinoza (1632–1677) were among his major sources of philosophical inspiration. When Cantor completed his *Habilitation*, for example, he chose a 'thesis' referring to the appendix of the first part of Spinoza's *Ethics*[1] for the public defense in order to get the license to teach at a university.[2] Subsequently, he continued to study Spinoza's works intensively.[3] On several occasions, Cantor mentioned Spinoza's philosophy in discussions with his colleagues and in his publications.[4] He called one of Spinoza's letters 'most important' (*höchst bedeutend*) and found it 'full of content' (*inhaltsvoll*).[5] He claimed that a correct and perspicuous theory of infinity lacks due to the insufficient development of Spinoza's and Leibniz's doctrines.[6] Once, Cantor even felt the necessity to defend himself against the suspicion of being a Spinozist.[7]

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[1] We used the Latin edition of Spinoza's *Ethica ordine geometrico demonstrata* (Spinoza, 1913), and the English translation (Spinoza, 2007). A collection of some of Cantor's papers is Cantor (1932b). For a complete bibliography of Cantor's works, see Tapp (2005), pp. 578–582.

[2] This 'thesis' reads: 'Iure Spinoza mathesi (Eth. pars. I. prop. XXXVI, app.) eam vim tribuit, ut hominibus norma et regula veri in omnibus rebus indagandi sit' ('Rightly Spinoza attributed such a power to mathematics (*Ethics*, first part, prop. 36, appendix) that for mankind, it is the norm and rule of the true in all things of research') See Cantor's *Habilitationsschrift 'De transformatione formarum ternarium quadraticarum'* (1869), reprinted in Cantor (1932b), p. 62. The translation is ours, as it will be in the following if nothing else is noted.

[3] The most reliable evidence is Cantor's notebook on the *Ethics*. Cantor began to study the *Ethics* in detail in 1871. He copied the text of Spinoza's definitions, axioms, and propositions onto the left-hand sides of the pages of a notebook and added his own summaries and commentaries on the right-hand sides. See the Appendix to this paper for further description, an edition, and translation of the main text of this notebook (the original is in Latin). As far as we can see, the influence of Spinoza in Cantor's thought has been studied only rudimentarily until now. Stauffer (1993) is primarily interested in interpreting Spinoza's philosophy with the aid of the modern, Cantorian conception of infinity.

[4] See, for example, Cantor (1932c [1883]), p. 175/(1996 [1883]), p. 890.

[5] Cantor mentioned the letter no. 29 (addressed to L. Meyer) ibid. In the place cited, Cantor also refers to Spinoza's *Cogitata metaphysica*.

[6] Cantor (1932c, 1996 [1883]), §5.

[7] At the end of the introduction to the 'Mitteilungen zur Lehre vom Transfiniten' Cantor (1932d [1887–1888]) discusses Cardinal Franzelin's (1816–1886) suspicion that Cantor himself was a Spinozist. In a letter to Kurd Laßwitz (1848–1910) dated 15 February 1884, Cantor seems to support Franzelin's suspicion, saying that the correct parts of Hegel's conception of infinity derive from Spinoza. However, at the end of the letter to Franzelin, which Cantor excluded from the publication in the 'Mitteilungen', he expresses his feeling that someday pantheism will be completely overcome. For a critial edition of the letter to Franzelin dated 22 January 1886, see Tapp (2005), pp. 268–271.

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For Cantor, mentioning Spinoza was not a side-affair or even a decoration of what mathematically matters. He was really concerned with the *philosophical* foundations or, at least, a *philosophical* embedding of mathematics in general and of set theory as the theory of infinite numbers in particular. But is there a true influence of Spinoza's philosophy onto Cantor's? Are there notional connections? If there are, what are the major parts of Spinozean theory which influenced Cantor in forming his own philosophical views? How does the metaphysical system of Spinoza go together with Cantor's efforts to work out a philosophical foundation for set theory?

This paper aims at providing a deeper insight into Cantor's views of infinity which range from infinite numbers to God. To this end, Section 2 examines traces of Spinozean thoughts which can be found in Cantor's works. An intimate connection which Cantor proposed between his transfinite numbers and Spinoza's infinite *modi* gives rise to draw a structural parallel between Cantorian and Spinozean notions in Section 4. As drawing this parallel requires some insights into Spinoza's theory, we outline his ontology, as far as it concerns the infinite, in Section 3. Section 5 is then devoted to another connection between Cantor's and Spinoza's thoughts under the keywords 'existence' and 'pantheism'. Section 6 provides some conclusions. The Appendix contains an edition and English translation of the hitherto unpublished text of a notebook from the *Cantor Nachlaß*, in which he copied and commented some passages of Spinoza's *Ethics* (see n. 3).

We want to take two critical lessons from historical methodology before we start to investigate the aforementioned questions. The one lesson is that apparent similarities in terminology do not necessarily indicate deep similarities of thought, especially in the case of two thinkers who are separated by two centuries. Only a broader similarity between whole areas of their theoretical networks allows inferring conceptual relations. The other lesson consists in the precaution not to believe naïvely in what the people under consideration themselves say about their being related to others. Concretely, Cantor's philosophical interests and his attempts to convince other scholars of his new theory *might* have led him to overemphasize Spinoza's influence. That this was not the case is a thesis that is supported by the results of our investigation.

2. Traces of Spinoza's influence in Cantor's conception of infinity

Cantor's philosophy of infinity starts with the distinction of actual and potential infinity—a philosophical distinction also familiar to the mathematicians of his time. But he went on downright philosophically asking for the relation of actual infinity to God. Are they to be identified? Is God *the* actual infinite? And is God's being *the* actual infinite the true reason for Aristotle's famous *infinitum actu non datur*—interpreted in the sense of *not being given in the world*?

For Cantor it is clear that the actual infinite—in 1883 he called it 'proper infinite' (*Eigentlich-Unendliches*)—must not be simply identified with God. Firstly, he told apart two kinds of actual infinity, namely God's infinity and non-divine infinity. Then, in 1885–1886, when he published *Über die verschiedenen Standpunkte in Bezug auf das aktuale Unendliche*,[8] he distinguished even three kinds of actual infinity, namely actual infinity *in concreto*, actual infinity *in abstracto*, and actual infinity *in Deo*. The actual infinite *in concreto* is what is realized if there are infinitely many concrete things. Cantor also calls it 'the transfinite'. The actual infinity *in abstracto* is realized by the transfinite numbers, and God's infinity is called 'absolute infinity'. For Cantor, this distinction gave rise to a classification of philosophical positions according to their admitting or denying of the three kinds of actual infinity.[9] Cantor counted himself in the group assuming the infinite in all three respects. Spinoza, in contrast, belongs to the group accepting the actual infinite *in concreto*, but refusing it *in abstracto*. In Cantor's eyes, Spinoza was an advocate of the existence of infinite quantities and nonetheless an opponent to the existence of infinite numbers.[10] Cantor (1932e [1886]) blames Spinoza for not distinguishing sufficiently between the absolute infinite and the transfinite, both being kinds of actual infinity. In his eyes, it was this mistake that led Spinoza to believe in the impossibility of actually infinite numbers and to assume infinite quantities which cannot be counted.

Indeed, Spinoza rejected the classical arguments against the existence of infinite quantities different from the Absolute, but he accepted the arguments against the existence of infinite numbers. To defeat traditional philosophical arguments against infinite numbers was one of Cantor's main motives for doing philosophy. So he felt the necessity to defeat Spinoza's conviction that actually infinite numbers are impossible. To that end he pointed out a *petitio principii* in Spinoza's arguments as they employ a notion of number in which finiteness is tacitly presupposed.

Despite this considerable criticism, Cantor was nonetheless convinced that a proper theory of the mathematical infinite depends on developing certain ideas of Spinoza's, especially the idea of *modus infinitus*. He held that it would be great scientific progress to bring out the exact relation between the finite and the infinite *modi* and to solve the difficult question how finite *modi* can have a being on their own and how they could possibly not be annulled by the infinite *modi*.[11] At this point, a closer look on Spinoza's theory is helpful (Sect. 3) before we further examine the role of infinite *modi* in Cantor's theory (Sect. 4).

3. Spinoza's conception of infinity

The *Ethics* contains Spinoza's most complete discussion of the concept of infinity. Although this work is called 'Ethics' for its primary goals and intentions, it is nevertheless a piece of theoretical philosophy and maybe called 'metaphysical'. In the following, some parts of Spinoza's theory are discussed which concern the notion of infinity.

Spinoza distinguishes three different senses of infinity: the sense in which it is said of God, the sense in which it is said of God's attributes, and the sense in which it is said of the *modi*.

In the *first sense*, infinity is predicated only of God or the *Deus sive natura*, that is, only of that being

> quod in se est et per se concipitur
> (which is in itself and is conceived through itself.)[12]

[8] Cantor (1932e [1886]), pp. 370–376. Cantor published three slightly different versions of 'Über die verschiedenen Standpunkte'. A first one appeared in a journal of the Swedish academy in 1885 under the title 'Über die verschiedenen Ansichten in Bezug auf die actualunendlichen Zahlen'. In 1886, a second version was published as 'Über die verschiedenen Standpunkte in Bezug auf das actuale Unendliche' in the *Zeitschrift für Philosophie und philosophische Kritik* and a third one called 'Über die verschiedenen Standpunkte in Bezug auf das Actualunendliche' in *Natur und Offenbarung*, a former German journal for interdisciplinary debates between natural sciences and theology. See Tapp (2005), pp. 580–581.

[9] Cantor (1932e [1886]), p. 373.

[10] Cantor (1932c [1883]), §5, pp. 175–177/(1996 [1883]), §5, pp. 890–892.

[11] In fact, the same question has to be asked concerning the infinite *modi* with regard to even 'stronger infinite' ones. See ibid., p. 892/177.

[12] Spinoza (1913, 2007), first part, def. 3.

However, 'being in itself' and 'being conceived through itself' are not identical. So one might place more stress on one than the other, that is, one might interpret Spinoza more *realistically* underlining that God is a simple being or one might interpret Spinoza more *idealistically* emphasizing that God is the being whose idea is simple.[13] We will come back to this alternative when we discuss the attributes in a minute. For Spinoza, *Deus sive natura* is however absolutely infinite, and that means infinite

> non autem in suo genere; quicquid enim in suo genere tantum infinitum est, infinita de eo attributa negare possumus; quod autem absolute infinitum est, ad ejus essentiam pertinet, quicquid essentiam exprimit et negationem nullam involvit.
> (not after its kind: for, of a thing infinite only after its kind, infinite attributes may be denied; but that which is absolutely infinite, contains in its essence whatever expresses reality, and involves no negation.)[14]

From an extensional point of view, God coincides with the totality of existing beings, both physical and mental. But from an intensional point of view, God is a whole which cannot be reduced to the sum of the single beings He extensionally comprises. The whole *natura* has a life of its own and this life is superstructured to the single lives of its parts.

Spinoza's *second sense* of infinity applies to attributes, more precisely, to the attributes of God. The notion of an attribute is defined by Spinoza in the following way:

> Per attributum intelligo id, quod intellectus de substantia percipit, tanquam ejusdem essentiam constituens.
> (By attribute, I mean that which the intellect perceives as constituting the essence of substance.)[15]

The attributes constitute the essence of the substance, that is, the essence of God. Here, this whole concept is put under the operator 'that which the intellect perceives as ...', and again the realistic/idealistic alternative arises. Do the attributes constitute the essence of the substance in reality or in perception? Although the quoted definition, strictly read, supports the idealistic view, there is also good evidence in favor of the realistic one. So, for example, when Spinoza says:

> Quo plus realitatis aut esse unaquaeque res habet, eo plura attributa ipsi competent.
> (The more reality or being a thing has, the greater the number of its attributes.)[16]

and

> Deus, sive substantia constans infinitis attributis, quorum unumquodque aeternam et infinitam essentiam exprimit, necessario existit.
> (God, or substance consisting of infinite attributes, of which each expresses eternal and infinite essentiality, necessarily exists.)[17]

Anyway, Cantor embraced the realistic interpretation and so, for the aims of this paper, there is no need to decide this widely discussed question.

Anyhow, Spinoza states explicitly in which sense God's attributes are infinite. He says:

> Per Deum intelligo ens absolute infinitum, hoc est, substantiam constantem infinitis attributis, quorum unumquodque aeternam et infinitam essentiam exprimit.
> (By God, I mean a being absolutely infinite—that is, a substance consisting in infinite attributes of which each expresses eternal and infinite essentiality.)[18]

So, there are two senses in which infinity is germane to God's attributes: they are infinite in essence and they are infinite in quantity.

The first sense (infinity in essence) might also be characterized as 'actually metrical' for the essence of the attributes is something more 'extended' than every single object included in it. This becomes more comprehensible when one recalls that, for Spinoza, every attribute expresses an infinite essence which is part of the infinite divine essence.[19] On the one hand, this implies that every attribute is a part of God's attributes. On the other hand, this implies that God's attributes are not reducible to other attributes. Furthermore, God's attributes cannot be divided for a division of one of God's attributes would lead to a division of the substance, which is unthinkable in the context of Spinoza's metaphysics. Metaphorically speaking, an attribute of God is like an 'organism' which arises out of its 'parts' but is yet not reducible to its 'parts' for it precedes or supervenes what it contains. This can be understood in analogy to the famous slogan that a whole can be more than the sum of its parts, and in particular to the view—most prominently expressed by Aristotle—that points are part of a straight line without the line being composed of the points. Spinoza makes this even clearer by considering the *extensio* as an example of God's attributes. Every possible natural object has a certain spatial extension. The *extensio* in general comprises all the possible particular extensions. Thus, the *extensio* is infinite in the 'metrical' sense. Like each other attribute of God, it is a *maximum in genere,* that is, an entity which belongs to a *genus*, but exceeds every other entity of this *genus.*

The second sense in which infinity applies to God's attributes (infinity in quantity), might also be called 'actually quantitative' for God's attributes form an actually infinite totality whose quantity exceeds every finite number. In short: God's attributes are infinitely many, that is, when one counted the single thoughts falling under the *cogitatio*, for example, this counting would never come to an end.[20] For Spinoza, however, this does not entail the existence of an infinite number. He opposed the general supposition that every quantity is measurable by a number. So he writes in his letter to Meyer in April 1663:

> Nam, praeterquam quod multa invenerunt, quae nullo Numero explicare possunt; quod satis numerorum defectum ad omnia determinandum patefacit; multa enim habent, quae nullo numero adaequari possunt, sed omnem, qui dari potest, numerum superat.
> (For, apart from that they [the mathematicians] have discovered many things not expressible by any number—which illuminates sufficiently the inadequacy of numbers for determining every-

[13] Cantor characterizes his own philosophical position as 'thoroughly realistic but, at the same time, no less idealistic' ('durchaus realistisch, zugleich aber nicht weniger idealistisch'). See Cantor (1996 [1883], §8, p. 896; 1932c [1883], §8, p. 181).
[14] Spinoza (1913, 2007), first part, explanation to def. 6.
[15] Ibid., def. 4.
[16] Ibid., prop. 9.
[17] Ibid., prop. 11.
[18] Ibid., def. 6.
[19] See, for example, ibid., notes to prop. 10 and prop. 11.
[20] One might say that a discrete *genus* is infinite in the sense that it can not be counted by a finite number, but only by an infinite number. But this is not a good way to put it because Spinoza explicitly denies the possibility of actually infinite numbers.

> thing—they know many things which can not be sufficiently expressed by any number because they exceed every possible number.)[21]

Thus, Spinoza held that *infinite* quantities do not have to be measurable by a number. Infinite numbers for him had something absurd, leading to contradictions like:

> ac propterea unus numerus infinitus erit duodecies major alio infinito.
> (one infinity would be twelve times as great as the other.)[22]

Apparently, Spinoza held infinite quantities and numbers as strictly apart as to that these contradictions do not transfer to infinite quantities.

The *third and final sense* in which infinity is used in Spinoza's *Ethics* is as applicable to *modi*. Spinoza defines *modi* as

> substantiae affectiones, sive id, quod in alio est, per quod etiam concipitur.
> (the modifications of substance, or that which exists in, and is conceived through, something other than itself.)[23]

Therefore the *modi* are not self-subsistent, but depend ontologically on something else—the substance. In a corollary to proposition 25, the notion of a *modus* is made clearer, when Spinoza says about the individual things that they

> nihil sunt, nisi Dei attributorum affectiones, sive modi, quibus Dei attributa certo et determinato modo exprimuntur.
> (are nothing but modifications of the attributes of God, or modes by which the attributes of God are expressed in a fixed and definite manner.)[24]

Thus, the finite *modi* are nothing but the individual creatures, minds, and thoughts (will, cupidity, love, and so on)—or their individual manifestations. For example, according to Spinoza, the human intellect is a *modus*. In a certain sense, the *modi* are just the phenomenal expressions of the attributes of God.

The infinite *modi* need a closer examination. The fundamental propositions concerning them are propositions 21 and 22 of the *Ethics*. Spinoza says:

> Omnia, quae ex absoluta natura alicujus attributi Dei sequuntur, semper et infinita existere debuerunt, sive per idem attributum aeterna et infinita sunt.
> (All things which follow from the absolute nature of any attribute of God must always exist and be infinite, or, in other words, are eternal and infinite through the said attribute.)[25]

Hence, the main difference between attributes and *modi* is that an attribute is in itself, while a *modus* is ontologically dependent on attributes for it exists only as something like a property of an attribute. For example, the *cogitatio* is a being in itself which includes every thinking being (even when the *cogitatio* is not the 'sum' of the thinking beings, but has its own proper 'life'), and God's infinite intellect is a *modus* of this *cogitatio*. This *modus* is infinite and eternal, but it is not a being in itself, and it includes only part of the *cogitatio*. Furthermore, the attributes belong to the *natura naturans*, while the *modi* belong to the *natura naturata*. The infinite intellect and the *cogitatio* are also different as the *cogitatio* is more extended having the intellect as one of its *modi*.[26] The first point can be clarified by considering an analogy: a man is a being in itself, but an arm or a leg are not beings in themselves, they do not have an essence which is independent from the one of the man. Furthermore God's intellect is not as extended as the *cogitatio* because, for example, it does not include God's infinite and eternal will, God's infinite and eternal love, and so on. These last are other infinite *modi*. In this way, Proposition 21 supplies Spinoza's first concept of a *modus* of God as directly deriving from the nature of God's attributes.

Proposition 22 presents Spinoza's second characterization of an infinite *modus*:

> Quicquid ex aliquo Dei attributo, quatenus modificatum est tali modificatione, quae et necessario et infinita per idem existit, sequitur, debet quoque et necessario et infinitum existere.
> (Whatsoever follows from any attribute of God, in so far as it is modified by a modification, which exists necessarily and as infinite, through the said attribute, must also exist necessarily and as infinite.)[27]

Those are 'infinite *modi* of second kind' because they do not derive directly from God's attributes, but from their modifications. Spinoza gave an example of an infinite *modus* of second kind in a letter to Schuller on 29 July 1675.[28] Here Spinoza calls the aspect (*facies*) of the universe an infinite *modus* of second kind. The aspect changes in infinite ways, but it remains always the same (as a person can change her position remaining the same person). Therefore the aspect of the universe is an infinite *modus,* which is a modification of the *extensio*.

So, in the *Ethics,* Spinoza distinguishes three senses of infinity: the infinity of God, the infinity of God's attributes, and the infinity of *modi*. The attribute and the modi case have in common that infinity means a *maximum in genere*. A common feature of all three senses of infinity is that they are actualizations of a potentially infinite process.

4. An analogy and its limits

Having surveyed a portion of Spinoza's ontology, we can now come back to Cantor's view that Spinoza admitted the actual infinite *in concreto*, but refused it *in abstracto*. This was indeed so, for Spinoza was convinced of the existence of the infinite attributes and *modi*, and so he can be said to have accepted the actual infinite *in concreto*. And, on the other hand, he refused it *in abstracto* for he did not believe in the existence of infinite numbers.

In contrast, Cantor held that actually infinite numbers are possible and that their type of infinity has to be especially distinguished from the absolute infinity of God. When he wrote to Rudolph Lipschitz (1832–1903)[29] in November 1883, Cantor underlined this point. Concerning the sort of actual infinity that can be treated mathematically by a theory of infinite numbers, he added a consideration that provides a basis for a structural analogy between Spinoza's theory and his own one. He said:

> Das Vollendetunendliche findet sich allerdings in gewissem Sinne in den Zahlen ω, $\omega + 1$, ... , ω^{ω}, ...; sie sind Zeichen für gewisse Modi des Vollendetunendlichen und weil daher das

[21] Spinoza to L. Meyer, 20 April 1663, see Spinoza (1914), p. 41.
[22] Spinoza (1913, 2007), first part, note to prop. 15.
[23] Ibid., def. 5.
[24] Ibid., cor. to prop. 25.
[25] Ibid., prop. 21.
[26] Ibid., note to prop. 29, prop. 31, and demonstration.
[27] Ibid., prop. 22.
[28] Spinoza (1914), pp. 205–206.
[29] Cf. Cantor's letter to Rudolph Lipschitz of 19 November 1883 in Lipschitz (1986), p. 33.

> Vollendetunendliche in verschiedenen, von einander mit der äussersten Schärfe durch den sogenannten 'endlichen, menschlichen Verstand' unterscheidbaren Modificationen auftreten kann, so sieht man hieraus deutlich wie weit man vom Ab-so-lu-ten entfernt ist.
> (The completed infinite can be found in a certain sense in the numbers $\omega, \omega+1, \ldots, \omega^{\omega}, \ldots$; they are signs for certain *modi* of the completed infinite. So the completed infinite can occur in different modifications which *can be* most sharply *distinguished* by the *so-called 'finite* human mind'. That shows clearly *how far from the absolute* we are.)

The word *Modus* is hard to understand if one takes the German text as it is. But as Cantor had the philosophy of Spinoza in his intellectual background, the passage may be read as referring to Spinoza's technical term *modus*. Further evidence for this reading is supplied by Cantor's statement that the development of a mathematical theory of the infinite hinges upon a further development of Spinoza's idea of *modus infinitus*, as was mentioned above.

The transfinite ordinal numbers ($\omega, \omega + 1, \ldots, \omega^{\omega}, \ldots$) are one of the most important inventions of Cantor's set theory. Transfinite numbers have the following two properties (among others):

(A) Each one is larger than every finite number.
(B) Each one can be represented by the set of its infinitely many (ordinal) predecessors.

These two properties can be interpreted in Spinozean terms. When the finite numbers are interpreted as finite *modi* then (A) means that every transfinite number represents a metrical infinite and (B) means that every transfinite number represents a quantitative infinite. So the transfinite numbers have both properties of Spinoza's infinite *modi*. Hence, it is plausible that Cantor used Spinoza's notion of *modus* in his assertion, that the numbers represent signs for the *modi* of the completed infinity.

Further evidence for this similarity between Spinoza's *modus* and Cantor's transfinite numbers can be found in Cantor's analysis of a classical objection to infinite numbers. In the *Grundlagen*, Cantor dealt with the argument that, if infinite numbers existed, they would annul the finite numbers. To answer this objection, Cantor's reference point was just the Spinozean theory of the finite and infinite *modi*. Cantor wrote:

> Ein besonders schwieriger Punkt in dem Systeme des Spinoza ist das Verhältnis der endlichen Modi zu den unendlichen Modis; es bleibt dort unaufgeklärt, wieso und unter welchen Umständen sich das Endliche gegenüber dem Unendlichen oder das Unendliche gegenüber dem noch stärker Unendlichen in seiner Selbständigkeit behaupten könne ... Ist ω die erste Zahl der zweiten Zahlenklasse, so hat man $1 + \omega = \omega$, dagegen ist $\omega + 1 = (\omega + 1)$, wo $(\omega + 1)$ eine von ω durchaus verschiedene Zahl ist. Auf die *Stellung* des Endlichen zum Unendlichen kommt also, wie man hier deutlich sieht, alles an; tritt das erstere vor, so geht es in dem Unendlichen auf und verschwindet darin; *bescheidet* es sich aber und nimmt seinen Platz *hinter* dem Unendlichen, so bleibt es erhalten und verbindet sich mit jenem zu einem neuen, weil modifizierten Unendlichen.
> (An especially difficult point in Spinoza's system is the relationship of the finite modes to the infinite one; it remains unexplained how and under what circumstances the finite can maintain its independence with respect to the infinite, or the infinite with respect to still higher infinities ... If ω is the first number of the second number class, then one has $1 + \omega = \omega$, but $\omega + 1 = (\omega + 1)$, where $(\omega + 1)$ is a number entirely distinct from ω. Therefore, as one here clearly sees, everything depends on the *placement* of the finite relative to the [in]finite; if the former comes first, it merges into the infinite and vanishes therein; but if it *contents* itself to take its place *after* the infinite it is preserved and unites with it to form a new, because modified, infinite.)[30]

Cantor held that a correct theory of infinity can take its origin in Spinoza and has to overcome the limits that Spinoza himself had imposed to his own theory. Cantor showed that the finite is not necessarily annulled by the infinite. The quotation clearly confirms that Cantor saw an interconnection between the transfinite numbers and the infinite *modi*.

This line of interpretation can be elongated. Spinoza's ontology has (at least) four major levels: the *modi* of second kind, the *modi* of first kind, the attributes of God and finally God himself, the absolute infinity. To a certain extent, there is a structural parallel to this Spinozean hierarchy in Cantor's set theory.

In the considerations above, we have left open both, whether we speak about transfinite ordinal or transfinite cardinal numbers, and whether we speak about infinite *modi* of first or of second kind. Both conditions, (A) and (B), are fulfilled by ordinals as well as cardinals. Can the two borderlines, the one between ordinals and cardinals, and the one between infinite *modi* of first and of second kind, be made coincide?

A decisive suggestion can be found again in Cantor's letter to Lipschitz, when Cantor says that the ordinal numbers are *modi* of the actual infinite and that in this way the completed infinite occurs in *modifications* conceivable by the finite human mind. What can it mean to speak of 'modifications' in the context of transfinite ordinal numbers? Transfinite ordinal numbers can indeed be conceived as following from modifications, namely, from modifications of a set by taking that to be a (re-) (well-) ordering of the set. For transfinite ordinals are the order types of (well-founded) sets, they can be said to follow from a set's modification in the sense of a specific well-ordering of it.[31] But is this interpretation of 'modifications' not too remote from Spinoza's views? We do not think so. Spinoza described the infinite *modi* of second kind as 'following in so far as it is modified by a modification' (*sequitur quatenus modificatum est tali modificatione*).[32] And this is true on the level of the transfinite ordinal numbers when they are said to stem from modifications of the ordering of an infinite set.

So, one can fix the first element of the structural analogy between Spinoza's and Cantor's philosophy of the infinite by taking the transfinite ordinal numbers to play the role of infinite *modi* of second kind. And in Cantor's theory an 'object' which has the same functional role as the *modi* of the first kind is directly at hand. In Spinoza, infinite *modi* of the second kind derive from a divine attribute by a modification, while the infinite *modi* of the first kind follow directly from a divine attribute, without any modification. Having interpreted the modifications as re-orderings of an infinite set, it seems just consistent to draw the parallel between the infinite *modi* of first kind and the cardinal numbers. In fact, the cardinal number of a set is not modified by any permutation of the elements of the set.

The next step in a parallelization of Cantor's and Spinoza's theory has to deal with the central concept in Cantor's theory—the concept of a set. For finding an analogue in Spinoza's theory, there are two options.

[30] Cantor (1932c [1883]), §5, p. 177/1996 [1883]), §5, p. 892.
[31] The natural ordering of the set of natural numbers 0, 1, 2, 3, ... has the transfinite ordinal number ω as its order type, while a *modified* ordering like 2, 3, 4, ..., 0, 1 has order type $\omega + 2 \neq \omega$.
[32] Spinoza (2007), prop. 22.

4.1. Sets as modi of third kind

One could try to interpret the sets as belonging to the same ontological level as the transfinite numbers (that is, as the *modi*). In this case, they must be considered as a new sort of *modi*, as *modi* of third kind.

A cardinal number is not modified by any change or any permutation of the elements. An ordinal number is not modified by any change of the elements, but it is modified by permutations of the elements. A set is modified both by a change of the elements and by a permutation of the elements (according to one of Cantor's two notions of set).[33] The difference between sets and *modi* of second kind can thus be that *modi* of second kind can be modified by one kind of modification, while sets can be modified by two kinds of modifications. Hence, it seems consistent to interpret sets as *modi* of third kind, more specific and modifiable in more ways than the ordinal and cardinal numbers. This interpretation goes along well with Cantor's abstraction account of transfinite numbers: one abstraction process leads from a set to its ordinal number, a second one from the ordinal number to its cardinal number.[34]

But on the other hand, the cardinal and the ordinal numbers are numbers of sets, they can be said to refer to sets or to be attached to sets. Therefore, it is doubtful whether numbers and sets really belong to the same ontological level. If numbers depend on the sets and express some of their important properties, then the ontological level of the sets shall be superior to the one connoting the numbers. So, we consider the following alternative.

4.2. Sets as attributes

According to this second option, sets are interpreted as attributes of God, while the transfinite numbers are *modi* of these attributes.

Sets and numbers are strictly connected and Cantor himself stresses their connection. In Cantor's eyes, a set has always a cardinal number and an ordinal number (or, at least, an order type) and vice versa every cardinal and ordinal number is associated to the set of its (ordinal) predecessors. (In modern terms, that comes down to the fact that Cantor held the Axiom of Choice to be true.) We read indeed:

> Unendliche Zahl und Menge sind unlösbar miteinander verknüpft; gibt man die eine auf, so hat man kein Recht mehr auf die andere.
> (Infinite number and set are indissolubly tied; if one abandons the former, one loses the right to the latter.)[35]

It seems to us that in Cantorian theory the sets are the real first basic element. This does not imply that a set can be conceived without its own cardinal number and without its order type; in this sense, sets and numbers are logically co-original. But the numbers represent properties of the sets. They are well-defined closed unities in themselves, but ontologically dependent from the sets—just like Spinoza's *modi* with respect to the attributes.

The second option (to take sets as attributes) seemingly avoids the problems of the interpretation as modi of third kind. But when we try to extend this interpretation to Cantor's notion of inconsistent multiplicities, this option faces other problems.

4.3. Absolute infinity and inconsistent multiplicities

In 1883 Cantor made the famous remark that the endless series of infinite numbers can be seen as an adequate symbol of the Absolute.[36] This thought is expressed more clearly and in more detail in the letters to Dedekind in July–August 1899.[37] While Burali-Forti, who had discovered the paradox of the biggest ordinal number, concluded that set theory leads to contradictions, Cantor gave a completely different interpretation which holds also for the paradoxes of the system of all cardinal numbers and of the system of all sets. For the proofs of the paradoxes make use of the principle that every totality can be considered as a set; the proofs are nothing but *reductiones ad absurdum* that this principle is false. There are multiplicities which lead to contradictions if they are taken as sets, as mathematical unities, and therefore can not be conceived as sets. Cantor called these totalities *inconsistent multiplicities*.

In an intuitive sense one could say that these inconsistent multiplicities are too big to be sets.[38] This gives rise to the aforementioned extension of our interpretation, namely, to take an inconsistent multiplicity of the kind mentioned to be a *maximum in genere*. All three examples—the multiplicity of all ordinal numbers, the one of all cardinal numbers, and the one of all sets—are the extensions of notions that can be conceived of as *genera*. So it is obvious to us to interpret the system of all ordinal numbers as the *maximum in genere* of the *genus* 'ordinal number', and accordingly for cardinals and sets.

Hence, it seems that those inconsistent multiplicities have a role which is quite similar to Spinoza's attributes of God. First of all, they are infinities which are a *maximum in genere*—they can not be increased. In a certain view of mathematics, it is this character which excludes the inconsistent multiplicities from the field of the 'objects' which can be treated mathematically while, on the contrary, ordinal numbers, cardinal numbers, and sets are mathematical entities which can always be increased and compared.[39] Furthermore, the Spinozean attributes contain all entities of a determined kind—for example the *cogitatio* contains all thoughts—as do the inconsistent multiplicities. Finally, we have seen that, in Spinoza's ontology, God's attributes represent the fundamental ontological essence of the substance and, therefore, can not be separated from the substance.

This interpretation may provide an important aspect in the understanding of Cantor's assertion that the absolutely infinite series of cardinal numbers represents a symbol of the Absolute. But it is difficult to insert the notion of set into this interpretative scheme. The problem is that, if the inconsistent multiplicities have the same ontological *status* as Spinoza's attributes, then the sets can not be called 'attributes' in the same sense. Sets are well determined mathematical unities and there is no contradiction in taking them as a whole. Sets are measured by numbers, while this is not the case for the inconsistent multiplicities.

Perhaps, one could try to tackle the problem of sets by taking them to be 'attributes of second kind', while the Spinozean attributes are 'attributes of first kind'. Or, the sets could be interpreted as *modi* of third kind, as was discussed above (Sect. 4.1). But in either way of interpreting Cantor's philosophy of the infinite in this 'Spinozistic' manner, it is necessary to introduce a new ontological level which does not exist in Spinoza's theory.

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[33] Cantor operates with two different conceptions of sets. Here, we rely on the conception according to which sets have an internal ordering. The other one, which is used in today's mathematics exclusively, is that the ordering is distinct from the set which is ordered; indeed, the ordering is itself a (different) set.

[34] See Cantor (1932a [1895–1897]), pp. 282, 297–298.

[35] Cantor (1932d [1887–1888]), p. 394.

[36] Cantor (1932f [1879–1884]), Nr. 5, p. 205.

[37] For these letters see Cantor (1932b), pp. 443–450; Ewald (1996), pp. 926–940.

[38] Cp. Hallett's view of set theory as limitation of size (Hallett, 1984).

[39] In his early works, Cantor took for granted that for two cardinal numbers a and b, either $a > b$, $a = b$, or $a < b$. Afterwards he realized that this trichotomy has to be proved. He attempted a proof in Cantor (1932a [1895–1897]), but a successful proof would have required to make use of the axiom of choice (AC).

To sum up: We tried to draw a parallel between Spinoza's and Cantor's theories of the infinite. The ordinal numbers are aligned with infinite *modi* of second kind, the cardinal numbers with the infinite *modi* of first kind, and some maximal inconsistent multiplicities with God's attributes. For the sets, there is no proper place in this schema, although they are indispensable in a Cantorian ontology of infinite numbers and inconsistent multiplicities. This shows the limit of our proposed parallelization of Cantor and Spinoza.

5. Existence and pantheism

Another connection between Cantor and Spinoza goes far beyond the structural parallel we discussed in the last section. It can be characterized by the keywords 'existence' and 'pantheism'.

5.1. Existence

In §8 of his 'Grundlagen einer allgemeinen Mannigfaltigkeitslehre' (Cantor, 1932c [1883]), Cantor makes some remarks about his elaborated view on the reality of mathematical entities. In general, he discerns two senses of existence, namely 'immanent' and 'transient' existence. The immanent reality of numbers is their reality as mental objects, as constituents of thoughts, and as instruments of the mind. The transient reality is somewhat more complicated. Cantor says that numbers have transient reality

> als sie für einen Ausdruck oder ein Abbild von Vorgängen und Beziehungen in der dem Intellekt gegenüberstehenden Außenwelt gehalten werden müssen.
> (to the extent that they must be taken as an expression or copy of the events and relationships in the external world which confronts the intellect.)[40]

The existence of classes of objects which can be counted by the abstract numbers is one kind of their (possible) transient reality.

The notion of immanent reality is related to Spinoza's term *adaequatus*.[41] Spinoza used this term to characterize true ideas independently from their relation to an object which is imagined by the idea.[42] So adequate ideas comprise all 'internal' or 'immanent' properties true ideas have, irrespectively of their relation to objects in an 'external' world.

Cantor held the thesis that—under certain circumstances with which we do not want to deal now—mathematical notions do have a transient reality, but that discovering this reality is an ambitious task of metaphysical research. In note 6 to §8, Cantor (1932c [1883]) points out that this immanent–transient connection unites him with Spinoza, as Spinoza issued the famous maxim:

> Ordo et connexio idearum idem est ac ordo et connexio rerum.
> (The order and connection of ideas is the same as the order and connection of things.)[43]

The reason for this close connection between immanent and transient reality is, for Cantor, to be seen in the

> Einheit des Alls, zu welchem wir selbst mitgehören.
> (unity of the all to which we ourselves belong.)[44]

5.2. Pantheism

Cantor's examination of Spinoza's philosophy is not limited to the problems concerning mathematics and its foundations. In Cantor (1932e [1886]), he asserts that the Achilles heel of Spinoza's *Ethics*—its pantheism—depends on the fact that Spinoza did not distinguish precisely between absolute and transfinite infinity, although he had recognized the existence of different actual infinities. And in fact, from a certain perspective, Spinoza's admitting of more than one actual infinity gives rise to an accusation of pan- and of polytheism.

To begin with the latter, one has to recall that Spinoza admitted the existence of three kinds of actual infinity: God, God's attributes, and God's infinite *modi*. Attributes and *modi* are infinite in a metrical and in a quantitative sense. The quantitative sense seems to be theologically unproblematic, for Christian belief goes well with assumptions such as that God has an infinite quantity of attributes, that God has created an infinite quantity of creatures, and so on. The problems arise with the metrical conception of actual infinity. To consider the example of the attributes: admitting that an attribute is a *maximum in genere* can be held to imply a substantialization of the attributes; then, in case of an infinite maximum, we would have another infinite substance. God would not be the only actually infinite substance and his attributes may appear as 'Gods' themselves—albeit Gods on a 'minor level'. This 'mistake of polytheism' is unacceptable from a theological point of view.

Similarly, pantheism is a stumbling block in Spinoza's theory. It consists in an identification of God, God's attributes, and God's *modi* with nature. Cantor thought that Spinoza's mistake depends on the fact that Spinoza did not provide a clear qualitative distinction between God and the other kinds of metrical infinity. In Cantor's opinion, the accusation of pantheism makes sense only because Spinoza had not clearly distinguished between an actual infinite which can be increased (transfinite) and an actual infinite which can not be increased (absolute). Thus, Cantor believed to have found the qualitative difference between actual infinites in the dichotomic pair increasable/non-increasable. According to this view, there is no problem anymore: the increasable actual infinite is not the Absolute; it cannot be confused with God. Once the transfinite and the Absolute are clearly distinguished and a mathematical treatment of the transfinite is made possible, doctrines which admit the existence of more than one actual infinity can not simply be identified with pantheism. The infinite *modi*, when characterized mathematically, were completely different from the Absolute. Hence, according to Cantor, there was no longer any danger of pantheism.

6. Conclusions

In Spinoza's philosophy, infinity plays a fundamental role. He was one of the main authors in Cantor's philosophico-theological background. But Spinoza did not construct a speculation on infinity as detailed as Cantor's. From Cantor's perspective this was a serious lack in Spinoza's theory. The fact that Spinoza did not believe in the existence of the infinite numbers was one reason for that.

This is not to mean that Cantorian set theory is a necessary development which departed directly from Spinoza. Freely, Cantor constructed a mathematical theory and, in the course of time, he felt more and more the need to justify it from a philosophical and a theological point of view. In contrast, Spinoza's aims were not mathematical at all. He wanted to construct *more geometrico* a 'system of the world' in which mathematics played no

[40] Cantor (1932c [1883], §8, p. 181; 1996 [1883], §8, p. 895).
[41] Cantor (1932c [1883], §8, p. 206, n. 5; 1996 [1883], §8, p. 918).
[42] Spinoza (1913, 2007), second part, def. 4 and explanation.
[43] Ibid., prop. 7.
[44] Cantor (1932c [1883], §8, p. 182; 1996 [1883], §8, p. 896).

important role, while in Cantor's thought, mathematics belongs to the core of a 'system of the world'. It seems to be consistent with Spinoza's philosophical views to admit the infinite in quantity but no infinite numbers, even if this is a mistake from Cantor's perspective. The fact that Cantor's philosophy rests—to a certain extent—on Spinoza must not hide the differences between the two thinkers.

Another reason for the lack of an advanced theory of infinity in Spinoza's thoughts are, however, the conceptual difficulties with which Spinoza scholars have to deal when they try to understand Spinoza. Historically one can sustain the thesis that one of the factors which made the successful invention of infinite numbers before Cantor impossible was that the concepts were not clear enough and contained too much non-explicated presuppositions.

A major difference between Cantor's and Spinoza's thoughts on infinity concerns background and context. Cantor presented a theory of infinity with two sides, the mathematical treatment of transfinite numbers and the philosophical discussion of the concept of infinity. The dialectic between the mathematical and the philosophical–theological aspect was continuously present in Cantor's work and it is important to be kept in mind in order to understand Cantor's philosophical attitudes in all their dimensions. Set theory was born in an environment which reached from mathematical things like Fourier series to philosophico-theological topics like the Absolute.

Spinoza's philosophy does not have a whole and complete theory of infinity, although it has many elements of it. The inspirational power of his theory was as big as to influence Cantor's invention of the first reliable mathematical theory of infinity. Moreover, it was—to a certain extent—possible to draw a parallel between Spinoza's general philosophy and Cantor's philosophy of set theory. Although this parallel clearly has its limits, we hold that there is more than a mere formal analogy between Cantor's thought and Spinoza's. Surely, Cantor was not a 'Spinozist' in a stricter sense. But his investigations of Spinoza's philosophy influenced deeply the philosophical conception of Cantor's main mathematical achievement—set theory.

Appendix: A Cantorian notebook on Spinoza's *Ethics*

The *Cantor Nachlaß* kept by the Staats- und Universitätsbibliothek Göttingen contains a sixty-two-page notebook in a blue card jacket (SUB Göttingen: Cod. Ms. G. Cantor 27). The entries start with the headline 'Ethica Benedicti de Spinoza.' and the remark 'Wintersemester 1871–1872.' Each page is divided into a left-hand and a right-hand side, with the left-hand sides dedicated to Latin quotations from Spinoza's *Ethica*. The right columns contain: on page 1 the table of contents of the *Ethica*:

1. Pars prima. De Deo. 1.
2. P[ars] s[ecunda]. De natura et origine mentis 25.
3. P[ars] t[ertia]. De origine et natura affectuum 49.
4. P[ars] q[uarta]. De servitute humana seu de affectuum viribus. 67.
5. P[ars] q[uinta]. De potentia intellectus seu de libertate humana. 87.

and then—as on the following pages—presumably Cantor's annotations to the text. This commentary is written with black ink while the Spinoza quotations are written with blue. Unfortunately, Cantor's main notes end with item 11 on page 7.

In the following, we present Cantor's notes in the original Latin version opposite to an English translation. The single paragraphs are preceded by remarks in square brackets which concern the arrangement of Cantor's notices. Passages Cantor underscored are reproduced in italics.

[diagonally below definition 1]

1. Unica solummodo est *substantia*, nempe subst. divina; quaecumque vero praeter unicam hanc substantiam existunt, non sunt nisi variae ejusdem *affectiones* sive *modi*: est autem substantia divina *aeterna* atque *infinita*; ejusque essentia formaliter identificatur tum cum ejus existentia tum etiam cum ejus potentia. P.I, pr. 7,8,14,15,34.

1. The substance is only one, that is the divine substance; but whatever exists besides this one substance, is nothing but different *affectiones* or *modi* of it: the divine substance is eternal and infinite; its essence is formally identified both with its existence and with its power. P.I, pr. 7,8,14,15,34.

[slightly below def. 2, alongside def. 3 and 4]

2. Quia autem quo plus realitatis alicui rei inest, eo plura attributa ei competunt, (P. I pr. 9) *substantia absoluta infinitis constat attributis*. (I, pr. 11). Porro haec attributa non procedunt ex efficientia substantiae absolutae, nec proinde sunt ipsa modi aliqui qui sint in hac substantia tamquam in alio, verum eorum unumquodque essentiam absolutae substantiae constituit atque exprimit. (I def. 4 et 6 et pr. 11). Ex quo fit tum ut unumquodque attributum sit aeternum (I pr. 19) et *in suo genere* infinitum (I def. 6 et demonstr. pr. 16), tum ut ipsum concipiatur *per se*, hoc est independenter a conceptu alius cujusvis attributi (I. pr. 10) [1/2]

2. But since the more reality is in a thing, the more attributes come together in it, (P. I pr. 9) the absolute substance is composed of infinite attributes. (I, pr. 11). Furthermore, these attributes do not spring from the efficiency (power) of the absolute substance, and consequently they are not *modi* which are in this substance as if they were in something different, but truly every one of them constitutes and expresses the essence of the absolute substance. (I def. 4 and 6 and pr. 11) From this it follows firstly that every attribute is eternal (I pr. 19) and infinite in its kind (I def. 6 and demonstr. pr. 16), and secondly that it is conceived through itself, i.e., independently from the concept of any other attribute (I. pr. 10)
[1/2]

Porro ex infinitis Dei attributis duo solummodo nobis perspecta sunt: infinita nempe atque *absoluta cogitatio* et infinita atque *absoluta extensio*. (II pr. 1 et 2). Ratio hujus est quod omnia *singularia* quae percipimus, sunt vel corpora, et sic determinati modi absolutae extensionis, vel sunt finiti modi cogitationis, et sic determinati modi absolutae cogitationis. (II ax. 5).

In addition, only two among the infinite attributes of God are clearly perceived by us: surely the infinite and absolute *cogitatio* and the infinite and absolute *extensio*. (II pr. 1 and 2). The reason of this is that all the single things which we perceive are either bodies, and hence determined *modi* of the absolute *extensio*, or they are finite *modi* of the *cogitatio*, and thus determined *modi* of the absolute *cogitatio*. (II ax. 5).

[alongside def. 5 and 6]

3. Quia autem ex infinito infinita infinitis modis sequi necesse est (I, pr. 16), hinc ex unoquoque attributo divino sequuntur infiniti *modi*, quorum unusquisque est quidem aeternus et in suo genere infinitus (I, pr. 21), qui tamen non constituunt essentiam substantiae, sed sunt in substantia et efficienter procedunt e substantia. (I sch. pr. 29). Quae etiam ratio est cur non dicantur attributa sed *modi* attributorum sive substantiae.

Rursum, eadem de causa, ex modo uniuscujusque attributi efficienter procedunt *infiniti modi modorum*, qui et ipsi sunt in suo genere infiniti et aeterni (I pr. 22). Itidem ex hisce modis modorum sequuntur infiniti modi alii, atque ex hisce rursum modi alii, et sic in infinitum, facto progressu a modis inderterminatioribus ad modos determinatiores.

Talis porro aeternae ac progressivae evolutionis substantiae divinae in modos usque quidem determinatiores, sed non [2/3] determinatos sive finitos, non est possibile figere terminum aliquem. Ex una quippe parte ab unoquoque modo utpote exprimente potentiam divinam, necesse est aliquid efficienter procedere (I, pr. 36); ex alia vero parte ab attributis vel a modis qui in suo genere sint infiniti atque aeterni, non est possibile procedere aliquid quod in suo genere sit finitum ac temporale (I pr. 21, 22).

3. For it is necessary that infinite things follow from the infinite in infinite ways (I, pr. 16), infinite *modi* follow from each divine attribute. Each one of these *modi* is eternal and infinite in its kind (I, pr. 21); they, however, do not constitute the essence of the substance, but they are in the substance and spring from the substance efficiently (I note to pr. 29). This is also the reason why they are not called attributes, but *modi* of the attributes or of the substance.

Once again, for the same reason, infinite *modi modorum* (infinite modes of modes) spring from a *modus* of every attribute, and these infinite modi are themselves infinite and eternal in their kind (I, pr. 22). In the same way, other infinite *modi* follow from these *modi modorum*, and from these again other modes, and so on *in infinitum*, proceeding from the more undetermined *modi* to the more determined *modi*.

Furthermore, it is impossible to determine any limit of such an eternal and progressive evolution of the divine substance in the *modi* until the more determined ones, but not [2/3] until the determined or finite *modi*. On the one hand, it is necessary that something really springs from each *modus*, since it expresses the divine power (I, pr. 36); on the other hand, it is impossible that something finite and temporal in its kind springs from attributes or *modi* which are infinite and eternal in their kind (I, pr. 21,22).

[alongside def. 7]

4. Hinc singulare quodcumque, hoc est, res quaevis, quae finita est et temporalis determinatur a causa quae et ipsa finita est et temporalis, et haec rursum ab alia ejusdemmodi, et sic in infinitum. Sive, quod idem est, quilibet modus substantiae divinae omnino *finitus* et *determinatus* non procedit ab attributo aliquo divino, nisi quatenus modificatum est modificatione quae et ipsa omnino finita sit et determinata; et haec rursum modificatio debet procedere ab alio modo omnino finito, et sic semper, quin possibile sit devenire ad primum in serie horum modorum finitorum (I, pr. 28).

4. Hence every individual thing, or everything which is finite and temporal, is determined by a cause which is also finite and temporal, and likewise this cause again by another one, and so on *in infinitum*. Or, which is the same, every entirely finite and determined *modus* of the divine substance does not spring from any divine attribute if not in so far as this attribute is modified by some modification which is finite and determined; and this modification must spring once again from another entirely finite *modus*, and so on, without being possible to reach the first one in the series of these finite *modi* (I, pr. 28).

[alongside def. 8]

5. Absoluta substantia atque infinita attributa quibus constituitur et exprimitur, sive aliter Deus, secundum quod per se concipitur et est causa omnium quae fiunt, dicitur *natura naturans*. Omnes vero modi attributorum, sive infiniti sint, sive finiti, secundum quod in Deo esse et ab efficiente Dei virtute procedere intelliguntur, dicuntur *natura naturata*. (sch. pr. 29)
[3/4]

Ut tamen patet ex praedictis, non funditus inter se differunt natura naturans et natura naturata, verum ipsae sunt una eademque substantia atque natura, quae secundum est natura naturans sese active evolvit in naturam naturatam.

Et quidem secundum quod ad naturam naturatam pertinent modi illi aeterni atque in suo genere infiniti qui infinita serie sive ex attributis divinis sive alii ex aliis procedunt, evolutio naturae naturantis in naturam naturatam stricto sensu est *aeterna*, hoc est, ipsa est simultanea, tota simul atque immobilis. Secundum quod vero ad naturam naturatam pertinet infinita series modorum finitorum sive rerum singularium, evolutio naturae naturantis in naturam [*originaliter*naturum] naturatam est quidem sempiterna, *temporaria* tamen, successiva atque mobilis.

5. The absolute substance and the infinite attributes by which it is constituted and expressed, or God, in so far as he is conceived in himself and is the reason for everything that happens, is called *natura naturans*. But all the *modi* of the attributes, both infinite and finite, in so far as they are conceived as being in God and proceeding from God's real virtue, are called *natura naturata.* (note to pr. 29)
[3/4]

As, however, it is evident from what was said above, the *natura naturans* and *natura naturata* are not completely different, but they are one and the same substance and nature, which in so far as it is the *natura naturans*, it actively develops into the *natura naturata*.

And in so far as those eternal and in their kind infinite *modi*, which spring from an infinite series either of divine attributes or of some of these and others from other *modi*, belong to the *natura naturata*, the evolution of the *natura naturans* into the *natura naturata* is eternal in a strict sense, that is, it is simultaneous, totally present and immoveable. But in so far as the infinite series of finite *modi* or of single things belongs to the *natura naturata*, the evolution of the *natura naturans* into the *natura naturata* is everlasting, nevertheless temporal, successive and moveable.

[alongside axiom 1]

6. Verumtamen omnia quae modo quocumque fiunt, sive sint modi aeterni et in suo genere infiniti, sive sint *singularia*, hoc est modi transitorii atque finiti; omnia haec, inquam, *ex absoluta necessitate divinae naturae* determinata sunt et ad existendum, quando et quamdiu existunt, et ad operandum, quando et quamdiu operantur (I, pr. 29). Quamobrem in rerum natura nullum datur *contingens*, neque possibile fuit res alio modo alioque ordine a Deo produci atque ipsae revera productae [4/5] sunt (I, pr. 29 et 33 et sch. 2 pr. 33).

Ratio hujus est—tum a) quod, ut dictum est, ex infinito infinita infinitis modis, hoc est omnia possibilia, sequi necesse est—tum b) quod essentia naturae divinae identificatur cum potentia divina, quodque subinde sicuti natura divina per essentiam suam est determinata ad existendum, sic et per essentiam suam ipsa est determinata ad operandum quidquid ipsi est possibile operari (I, sch. pr. 17 et pr. 34, 35)

6. Anyway all things which in whatever way come into being, whether they are eternal and in their kind infinite *modi*, or singulars, that is transitory and finite *modi*; all these things, I say, are determined to exist, when and how long they exist, and to operate, when and how long they operate, by the absolute necessity of the divine nature (I, pr. 29).

For this reason, there is nothing contingent in nature, neither was it possible that God produced the things in another way and in another order he has really produced them (I, pr. 29 and 33 and note 2 to pr. 33). [4/5]

The reason of this is—a) that, as it was said, it is necessary that infinite things follow from the infinite in infinite ways, that is every possible thing—b) that the essence of the divine nature is identified with the divine power, and that as the divine nature is determined to exist by its essence, so it is also by its essence determined to do everything which is possible for it to do (I note to pr. 17 and pr. 34, 35)

[alongside axiom 2]

7. Hinc Deus est *causa libera*, non quod ipse non absoluta necessitate operetur, sed quia ipse ex sola suae naturae determinatione et a *nemine coactus* operatur (I def. 7, pr. 17); aliae vero omnes res sunt causae necessariae sive coactae, quia ab alio determinantur ad operandum. (I def. 7, pr. 17).

7. Therefore God is *free cause*, not because he does not act by absolute necessity, but because he acts only by the determination of his nature and not constrained by anyone (I def. 7, pr. 17); but all the other things are necessary or constrained causes, since they are determined to operate by something else. (I def. 7, pr. 17).

[alongside axiom 3]

8. Hinc non operatur Deus propter fines quosdam; nec est in rerum natura intrinsecum discrimen inter bonum et malum, meritum et peccatum, pulchrum et deforme. Quippe, uti dictum est, etiam quae vulgo dicuntur mala et peccaminosa et deformia, ex infinita perfectione naturae divinae eo modo quo eveniunt necesse est evenire. Hac proinde atque aliae ejusmodi notiones obortae sunt ex praejudiciis hominum veras rerum causas ignorantium atque imaginantium Deum, sicut ipsos, propter utilitatem aliquam suam operari. (I app. et IV praef.) [5/6]

8. Hence God does not act for any purposes; and there is no inner distinction between good and bad, merit and sin, beauty and ugliness in the nature of things. Therefore, as said, even those things which are commonly called bad, sinful and ugly, must emerge from the infinite perfection of the divine nature in the very way in which they emerge. These and other such notions arose from the prejudices of men who ignore the true causes of the things and who imagine God acting for some own advantage, like they do. (I app. and IV pref.) [5/6]

[alongside axioms 4 and 5]

9. Porro evolutio substantiae quatenus ipsa est absoluta cogitatio, eodem ordine progreditur ac ejus evolutio quatenus ipsa est absoluta extensio, ac generatim quatenus ipsa concipitur sub alio suo attributo quocumque (II, pr. 7, sch.). Cujus ratio est quod per omne attributum absolutae substantiae una eademque ejus infinita essentia ac proin una eademque ei insita lex sese evolvendi exprimitur.

Hinc tum omni modo extensionis respondet modus aliquis cogitationis ei omni ex parte commensuratus—tum ordo et connexio idearum idem est atque ordo et connexio rerum. (II, pr. 7)

9. Furthermore the evolution of the substance in so far as it is the absolute *cogitatio* proceeds in the same order as its evolution in so far as it is the absolute *extensio* and generally in so far as it is conceived under whatsoever other one of its attributes (II, pr. 7, note). The reason for this is that by each attribute of the absolute substance its one and the same infinite essence is expressed, and thus also one and the same law of self-evolution inherent in it.

Here, as to every *modus* of the *extensio* there corresponds a *modus* of the *cogitatio* which is in every part commensurate to it—the order and connection of ideas is the same as the order and connection of things. (II, pr. 7)

[alongside axiom 6]

10. Immo sicuti substantia extensa et substantia cogitans una eademque sunt substantia, quae jam sub hoc jam sub illo attributo apprehenditur, sic etiam modus aliquis extensionis et idea illius modi *una eademque sunt res* sed duobus modis expressa. Ex. gr. circulus in natura existens et idea illius circuli, si quales in se sunt considerentur, una eademque sunt res duos veluti diversos aspectus induens. Hujus ratio est quod rerum prouti in se sunt, Deus revera est causa, non prouti unico aliquo sed prouti infinitis attributis constat. (II. sch. pr. 7). [6/7]

10. Like the extended substance and the thinking substance are one and the same substance comprehended now through one attribute and now through the other; so, also, a *modus* of the *extensio* and the idea of that *modus* are also one and the same thing, though expressed in two ways. For example, a circle existing in nature and its idea, when considered in themselves, are one and the same thing presenting two different aspects. The reason of this is that of things as they are in themselves God is the true cause not inasmuch as he consists of only one attribute but inasmuch as he consists of infinite attributes. (II, note to pr. 7) [6/7]

[alongside axiom 7]

11. Idcirio objectum ideae humanam mentem in suo *esse* constituentis est corpus aliquod singulare, illud videlicet quod homo unusquisque corpus suum proprium nominat. (II, pr. 13). Nec solum corpora hominum atque animantium sed corpora etiam vegetantium et mineralium animata sunt anima intelligente, gradibus tamen diversis. Videlicet quo corpus aliquod est reliquis aptius ad plura simul agendum et patiendum, eo etiam anima sive idea illius corporis est reliquis aptior ad *plura simul percipiendum*. Et quo corporis alicujus actiones magis ab ipso solo pendent, sive quo minus corpora alia cum eodem in agendo concurrunt, eo ejus mens aptior est ad distincte intelligendum.

Ex quibus patet quae causa sit cur mens humana reliquis praestet. (II sch. pr. 7). Ut tamen dictum est, tota natura rerum animata est, ipsaque *absoluta cogitatio* est veluti anima mundi totius.

11. Therefore, the object of the idea constituting the human mind in its being is a certain singular body, namely the one that every man calls his own body. (II, pr. 13). Not only are the bodies of the men and of the animals animated by an intelligent soul, but also the ones of the plants and of the minerals, however in different grades. Obviously, the more than others a body is able to act on and to suffer from several things at the same time, the more that others also the soul or the idea of that body is able to perceive several things at the same time. And the more the actions of a body depend only upon this body, or the less other bodies compete with it in its actions, the more its mind is able to understand distinctly.

From these considerations, it is evident what the cause is of the human mind being superior to others. (II, note to pr. 7). Nevertheless, as was said above, the whole nature of things is animated and the absolute *cogitatio* is quasi the soul of the whole world.

[From the next page (no. 11) on, Cantor copied Spinoza's propositions without further annotations.]

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